Teichmüller Basics

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Contents

1 The usual warning

Heads up! These are lecture notes from a course, and they come with no warranty. If you find errors, please write to moon.duchin@tufts.edu.

2 Classification of $\mathrm{Isom}^+(\mathbb{H})$

We have that $\mathrm{Isom}^+(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$ where $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \left(z \mapsto \frac{az+b}{cz+d} \right) \tag{1}$$

Note: Any isometry of $\mathbb{H}$ maps geodesics to geodesics.

We seek to classify the action of $\mathrm{PSL}_2(\mathbb{R})$ via its possible fixed points. This gives us 3 possible cases.

Case 1: There is one fixed point on the interior of $\mathbb{H}$. In this case the element of $\mathrm{PSL}_2(\mathbb{R})$ is called *elliptic*. The map acts by rotation of the disc model and we may conjugate the fixed point to the point 0 on the disc model since $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$. The map is then of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and is completely determined by angle of rotation. Thus there cannot be any other fixed points.

Case 2: There are two distinct fixed points on $\partial\mathbb{H}$. In this case the map is called *hyperbolic* and is of the form $\begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix}$. Hyperbolic maps are

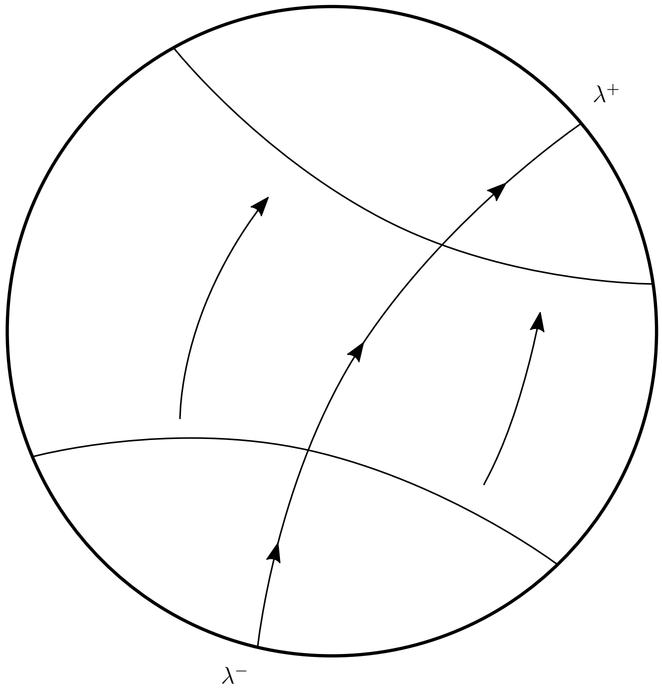

Figure 1: Example of a Hyperbolic Action with Attracting Fixed Point λ^+ and Repelling Fixed Point λ^-

characterized by having an attracting and a repelling fixed point on the boundary. The geodesic connecting these two fixed points are called the axis of the map and points on this geodesic are translated along this axis away from the repelling fixed point towards the attracting fixed point. Other geodesics in $\mathbb{H}$ are translated along the axis away from the repelling fixed point and towards the attracting one as well. See Figure 1 for an illustration of this.

Case 3: There is a single fixed point on $\partial\mathbb{H}$. In this case the map is called *parabolic* and is of the form $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. Parabolic elements act by translating along horocycles.

These 3 types of maps completely classify the elements of $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{H}$.

Proposition 1. *Hyperbolic geometry has no rectangles.*

Proof. We have a proof by picture with Figure 2. Taking the imaginary axis

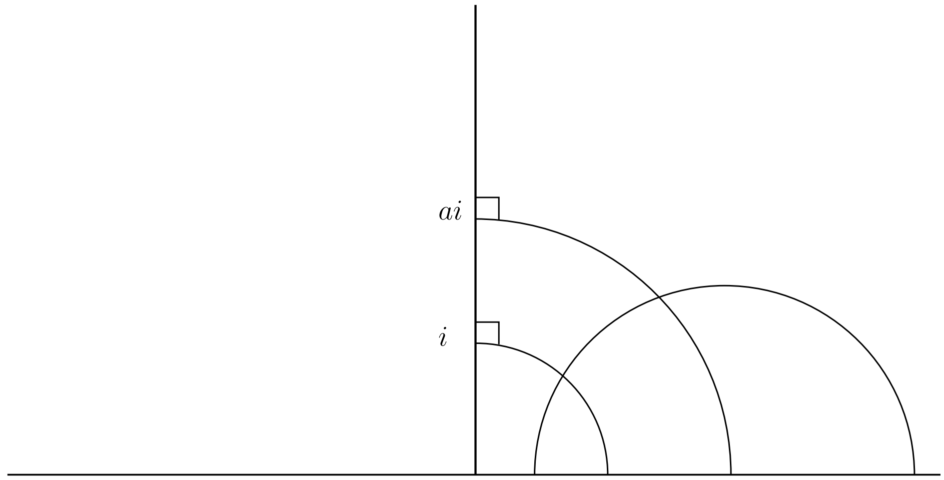

Figure 2: Proof of no Hyperbolic Rectangles

as one side of our possible rectangle and given two circles intersecting the imaginary axis at right angles there is no third circle which will intersect both perpendicular. □

Proposition 2. *Commuting elements of Isom*$^+(\mathbb{H})$ *preserve each others fixed point sets.*

Proof. Let $A, B \in \text{Isom}^+(\mathbb{H})$ such that $AB = BA$. Set $F_A :=$ fixed point set of A. Let $x \in F_A$ (so that $Ax = x$) and say $Bx = y$. Then we have

$$Ay = ABx = BAx = Bx = y \tag{2}$$

Therefore we have that $y \in F_A$. □

2.1 Pants

Pants are surfaces with genus 0 and three boundary components, i.e. $S_{0,3}$. We can uniformly geometric a pair of pants as a doubled right-angled hyperbolic hexagon. See Figure 3.

Exercise 1. *Prove that a right-angled hyperbolic hexagon is determined by the length of 3 alternating sides. Thus we can conclude that the boundary lengths of pants determine their geometry.*

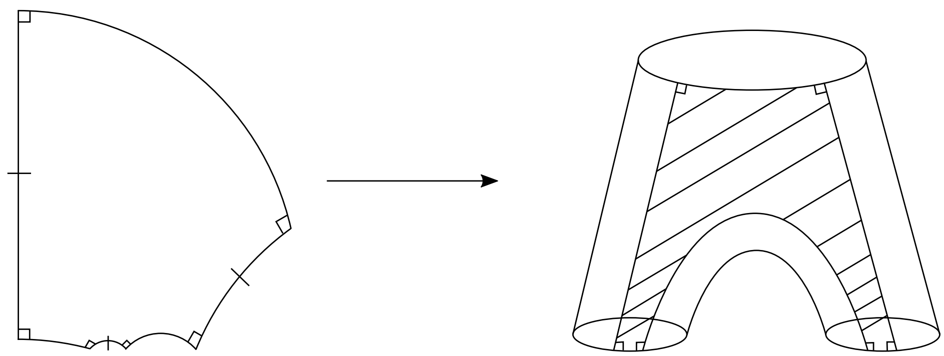

Figure 3: Pants Realized as Doubled Right-Angled Hyperbolic Hexagon

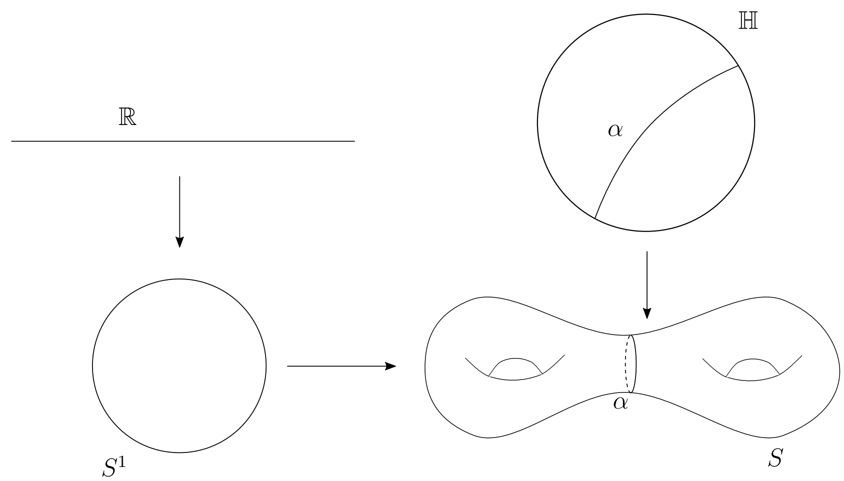

Figure 4: Lift of a Simple Closed Curve α to the Universal Cover of S

3 Simple Closed Curves

Definition 3. *Let S be a surface. A* ***simple closed curve (SCC)*** *is an embedding $S^1 \to S$. We denote the set of simple closed curves on a surface as $\mathbb{S}$.*

Note: Curves are only considered up to homotopy.

We can examine the lift of a simple closed curve $\alpha \in \mathbb{S}$ to the universal cover of S, $\tilde{S} = \mathbb{H}$, to see that the lift of α has a pair of endpoints on $\partial\mathbb{H}$ as in Figure 4.

Proposition 4. *For a hyperbolic surface S, any essential closed curve is* *homotopic to a unique geodesic.*

Proof. We can construct the unique geodesic. Any lift of some curve has two endpoints on $\partial\mathbb{H}$ which are connected by a unique geodesic. Now we can build the homotopy via projection from the curve onto this geodesic. Note that homotopy preserves endpoints on $\tilde{S}$. $\square$

Definition 5. *Let $\alpha, \beta \in \mathcal{S}$. The* ***(geometric) intersection number*** *of α and β is defined to be the minimum number of intersections of any homotopy representatives of α and β. It is denoted by $i(\alpha, \beta)$.*

This definition raises the question of how do we know when two simple closed curves are in minimal position? To answer this question we make the following definition.

Definition 6. *Two simple closed curves α and β in S form a* ***bigon*** *if there is an embedded disk in S with boundary the union of an arc of α and an arc of β which intersect in only two points.*

Now we can state the following proposition.

Proposition 7. *(Bigon Criterion) Two simple closed curves α and β in a surface S are in minimal position if and only if they do not form any bigons.*

Proof. (Sketch) If we suppose that α and β have no bigons then any pair of lifts intersect at most once. We give a sketch of the "innermost disk" argument. Suppose that they intersect in at least two points. Then there is an embedded disc $\tilde{D}$ in $\tilde{S}$ bounded by an arc $\tilde{\alpha_1}$ of $\tilde{\alpha}$ and an arc $\tilde{\beta_1}$ of $\tilde{\beta}$ as in Figure 5. Thus, downstairs in S, we have ∂D embedded in S and so the region can be eliminated via homotopy which reduces the minimum intersection. $\square$

4 Geometric Complex Analysis

Definition 8. *Let $f(z)$ be a complex function. Define the* ***complex derivative*** *of $f(z)$ to be*

$$f'(z) := \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{3}$$

We say that f is ***holomorphic*** *or* ***complex analytic*** *if its complex derivative exists.*

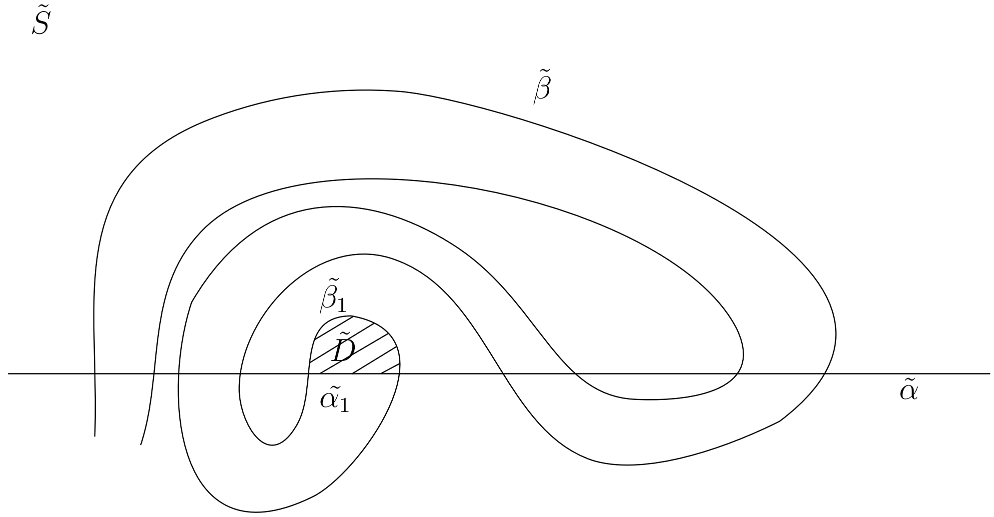

Figure 5: Innermost Disk Argument

Recall that we have a vector space isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$ as follows:

$$z = a + bi \longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \tag{4}$$

We can also represent multiplication by $z = a+bi$ as a linear transformation of $\mathbb{R}^2$. It is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (a^2 + b^2)$ where θ is the argument of z. Therefore, we see that complex multiplication is scaling composed with rotation.

Examples:

- $f(z) = z$, $f'(z) = 1$
- $f(z) = \bar{z}$ is not differentiable.

Note: One can show that $f'(z) = 0 \Rightarrow f(z)$ is constant.

When we think of $f(z)$ as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ we can formulate conditions known as the Cauchy-Riemann equations to check whether f is complex-differentiable. If we write $f(x+iy) = u + iv$ then its Jacobian is of the form $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$.

Theorem 9 (Cauchy-Riemann Equations)**.** *$f(x+iy) = u + iv$ is complex-differentiable if and only if it satisfies the following two equations:*

$$u_x = v_y \tag{5}$$

$$v_x = -u_y \tag{6}$$

Example: Consider the above example $f(z) = \bar{z}$. We can rewrite this map as $f(x+iy) = x - iy$. Now we see that its Jacobian, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, is not of the right form and f does not satisfy the Cauchy-Riemann equations and thus is not complex-differentiable.

Definition 10. *f is **conformal** if it preserves signed angles between tangent vectors.*

Theorem 11. *f is conformal in a neighborhood if and only if f is analytic and $f' \neq 0$.*

Matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a^2+b^2 \neq 0$ and $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$, $c^2+d^2 \neq 0$ are conformal and anti-conformal respectively.

Exercise 2. *Show that any $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ can be written uniquely as a sum of a conformal and an anti-conformal matrix if not all of α, β, γ, and δ are zero. I.e. one can write $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & d \\ d & -c \end{pmatrix}$ uniquely.*

We now introduce a whole class of examples.

Definition 12. *A **Möbius Transformation** is a complex map of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Möbius transformations are in direct bijection with $PSL_2(\mathbb{C})$; $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \frac{az+b}{cz+d}$.*

*Similarly, a **Fractional Linear Transformation (FLT)** is a complex map of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. In this case there is a bijection between FLT's and $PSL_2(\mathbb{R})$.*

Exercise 3. *For $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ compute $f'(z)$, check if it is conformal, and interpret this map geometrically. Do the same with $a, b, c, d \in C$.*

Although we defined analyticity above in terms of differentiability there are in fact three equivalent definitions. That is, $f(z)$ is analytic if one of the following three conditions is satisfied:

(i) $f(z)$ satisfies that limit definition of $f'(z)$, i.e. it satisfies the Cauchy-Riemann equations,

(ii) $f(z)$ admits a power series expansion in a disk,

(iii) or $f(z)$ satisfies that Cauchy Integral Formula. That is, for $a \in \mathbb{C}$ and γ the boundary of a disk containing a we have

$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a} dz \quad \text{and} \tag{7}$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-a)^{n+1}} dz \tag{8}$$

We now state two classical results from complex analysis and give a proof of the second.

Theorem 13 (Liouville's Theorem)**.** *If f is analytic on $\mathbb{C}$ (entire) and bounded then f is constant.*

Theorem 14 (Maximum Modulus Principle)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected (domain). Then if f is analytic on Ω, the max value of $|f(z)|$ on $\bar{\Omega}$ occurs on $\partial\Omega$.*

Proof. Suppose not. Take $p \in \Omega$ with $|f(p)|$ maximal and f non-constant. Now without loss of generality we may assume that $|f(0)| = 1$ is maximal and that $|f(q)| < 1$ for some $|q| = 1$. By the Cauchy Integral Formula we have

$$1 = |f(0)| = \frac{1}{2\pi i} | \oint_C \frac{f(z)}{z} \tag{9}$$

$$\leq \frac{1}{2\pi i} \oint_C |f(z)| \tag{10}$$

$$< 1 \tag{11}$$

which is a contradiction. $\square$

Definition 15. *A **biholomorphism** is a holomorphic function with a holomorphic inverse. Denote the set of biholomorphic functions on $\Omega \subset \mathbb{C}$ as* $\mathrm{Bihol}(\Omega)$.

The set $\mathrm{Bihol}(\Omega)$ forms a group under composition. We list a few examples below.

Examples:

- $\mathrm{Bihol}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$
- $\mathrm{Bihol}(\mathbb{C}) = \mathrm{Aff} = \{Az + B | A, B \in C\}$
- $\mathrm{Bihol}(\hat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C})$

Exercise 4. *Prove the above three statements.*

4.1 Riemann Surfaces

Definition 16. *A **Riemann surface** is a 2-manifold with holomorphic tranisition functions. Alternatively, a **Riemann surface** is a conformal structure, i.e. a Riemannian metric modulo conformal equivalance.*

Examples:

- $\mathbb{C}$ with one chart $\phi(z) = z$.
- $\hat{\mathbb{C}}$ with 2 charts. $f_1(z) = z$ for $\hat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$ and $f_2(z) = \frac{1}{z}$ for $\hat{\mathbb{C}} \setminus \{0\}$.
- Flat torus with translation transition functions.

  Exercise 5. *Minimize the number of charts to get a translation surface on the flat torus.*

- Hyperbolic surfaces, $S_{g,n}, (\chi < 0)$ with charts given via Poincaré construction.
- Graph of $z \mapsto z^3$. Take $w = z^3$, then the graph is the set of points $\{(z, z^3 | z \in \hat{\mathbb{C}}\}$. Note that as z moves around the unit circle, w wraps around the unit circle three times, i.e. we can write $w = (re^{i\theta})^3 = r^3 e^{3i\theta}$ for $z = re^{i\theta}$. Note that we have $f'(z) = 3z^2$ so that the map is not conformal at 0.

Exercise 6. *Show that any analytic homeomorphism from $\mathbb{C} \to \mathbb{C}$ is affine.*

Next we state a number of results pertaining to Riemann surfaces and holomorphic maps and offer sketches of the proofs for some of them.

Proposition 17. *There does not exist a nonconstant holomorphic map from a compact Riemann surface S to $\mathbb{C}$.*

Proof. This follows from the maximum modulus principle. $\square$

Proposition 18. *There does not exist a nonconstant holomorphic map from $\mathbb{C}$ to a hyperbolic Riemann surface S.*

Proof. Let $f : \mathbb{C} \to S$ be holomorphic and consider the following diagram.

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "\pi"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & S \end{tikzcd}$$

Now note that $\mathbb{D}$ is bounded; apply Liouville's theorem. $\square$

Proposition 19. *Let S be a Riemann surface. If $\tilde{S} = \mathbb{C}$, then $\pi_1(S)$ is abelian.*

Proof. First note that $\mathbb{C}$ is a cover of S and $g \in \pi_1(S)$ acts by deck transformations on $\mathbb{C}$ and thus g has no fixed points. Also, $g : \mathbb{C} \to \mathbb{C}$ acts holomorphically. Therefore, g is affine, i.e. $g(z) = Az + B$. We can calculate any potential fixed points for g,

$$z = Az + B \tag{12}$$

$$z = \frac{B}{1 - A} \tag{13}$$

However, since g has no fixed points we conclude that $A = 1$ and that $g(z) = z + B$. Thus $\pi_1(S)$ is abelian. In fact, this shows that the only Riemann surfaces with $\tilde{S} = \mathbb{C}$ are the flat torus, annulus, or $\mathbb{C}$ itself. $\square$

Theorem 20 (Uniformization)**.** *Any simply connected Riemann surface is biholomorphic to $\mathbb{C}, \hat{\mathbb{C}}$, or $\mathbb{H}$.*

Theorem 21 (Riemann Mapping Theorem)**.** *Any homeomorphism from $\Omega \to \mathbb{D}$ can be upgraded to a biholomorphism. Alternatively, any homeomorphism of a Riemann surface, X, to $S^2, \mathbb{T}^2$, or $S_{g,n}$ can be upgraded to a biholomorphism from X to $S^2 = \hat{C}, \mathbb{C}/\Lambda$, or $\mathbb{H}/\Gamma$ where Λ is a discrete subgroup of $(\mathbb{C}, +)$ and Γ is a discrete subgroup of $PSL_2(\mathbb{R})$ (i.e., a Fuschsian group).*

Proposition 22. *$\mathbb{C} \setminus \{2 \text{ points}\}$ has an analytic cover by $\mathbb{H}$.*

Proof. It cannot be $\hat{C}$ because $\hat{C}$ is compact, and π_1 is not abelian so $\mathbb{H}$ is the only candidate left. $\square$

Theorem 23 (Small Picard Theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is a nonconstant holomorphic map, then $im(f) = \mathbb{C} \setminus$ at most 1 point.*

Now we can finally state the definition of Teichmüller space. In fact, we now define it in two equivalent ways.

Definition 24. ***Teichmüller space*** *is*

(i) $\mathcal{T}(S) := \{$*Riemann surface* $X \cong S\}/g : X_1 \to X_2$*, where g is a biholomorphic map isotopic to the identity, or*

(ii) $\mathcal{T}(S) := \{(X, f) | X$ *is a Riemann surface,* $f : S \to X$ *a homeomorphism*$\}$ *modulo the relation defined by the following commutative diagram*

$$\begin{tikzcd}
 & X_1 \arrow[dd] \\
S \arrow[ur, "f_1"] \arrow[dr, "f_2"'] \arrow[r, phantom, "\circlearrowleft"] & \\
 & X_2
\end{tikzcd}$$

5 Quasiconformal Maps

Given a complex function $f(x+iy) = u + iv$ we can compute its Jacobian $\text{Jac}(f) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. Any matrix is conformal if it is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and anti-conformal if it is of the form $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$.

Exercise 7. *Any non-0 matrix can be written uniquely into a sum of a conformal part and an anti-conformal part.*

f is formally a function of z and $\bar{z}$ so we can define its derivative with respect to z and $\bar{z}$, f_z and $f_{\bar{z}}$ allowing us to make the following definition.

Definition 25. *The* ***Beltrami differential*** *of a local diffeomorphism* $f : \mathbb{C} \to \mathbb{C}$ *(or* $f : X \to Y$ *for* X *and* Y *Riemann surfaces) is defined as*

$$\mu_f = \frac{f_{\bar{z}}}{f_z} \tag{14}$$

The motivation for this definition is that unlike conformal maps which send circle fields to circle fields, affine maps send circle fields to ellipse fields and thus we would like a way to measure the amount of dilatation (the ratio of the length of the major axis to the minor axis of an ellipse field) induced by an affine map. See figure 6 for an illustration of this.

Example: We will compute the Beltrami differential of the map $f(x + iy) = x + kiy, k > 1$.

$$f(z) = \frac{z + \bar{z}}{2} + ik\frac{z - \bar{z}}{2} = \frac{1 + k}{2}z + \frac{1 - k}{2}\bar{z} \tag{15}$$

$$f_z = \frac{1 + k}{2} \tag{16}$$

$$f_{\bar{z}} = \frac{1 - k}{2} \tag{17}$$

$$\mu_f = \frac{1 - k}{1 + k} \tag{18}$$

Definition 26. f *is* K***-quasiconformal*** *if*

$$\frac{1 + |\mu_f|}{1 - |\mu_f|} \leq K \tag{19}$$

almost everywhere.

Observations:

- $\mu_f = 0 \Leftrightarrow f_{\bar{z}} = 0 \Leftrightarrow f$ is conformal.
- $\mu_f = \infty$ for f anti-conformal.

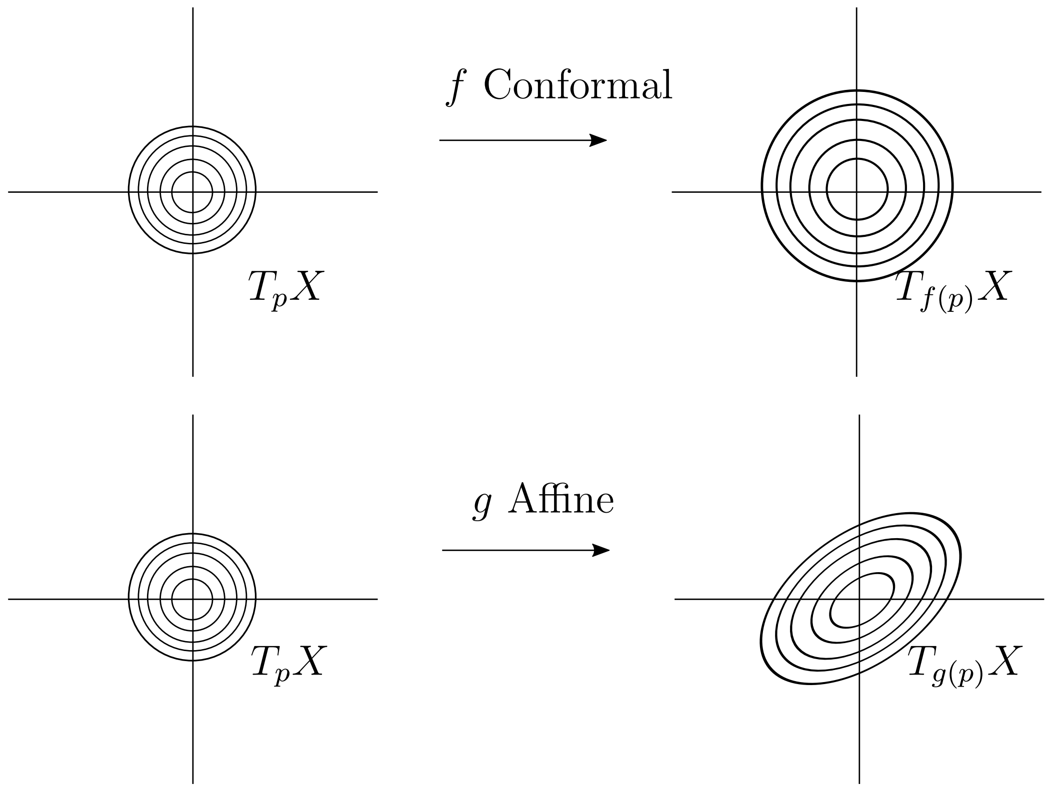

Figure 6: Circle Fields for Conformal vs. Affine Maps

- for $f(x+iy) = x + kiy$ we have $\frac{1+|\mu_f|}{1-|\mu_f|} = k$ and $\mu_f \to -1$ as $k \to \infty$ (see example above).

Exercise 8. *(1) For $f : \mathbb{C} \to \mathbb{C}$ a diffeomorphism at $p \in \mathbb{C}$ check that*

(a) At $(p, v) \in T_p X$ we have

$$\frac{\max_{|v|=1} |f_*(v)|}{\min_{|v|=1} |f_*(v)|} = \frac{1+|\mu_f|}{1-|\mu_f|} \tag{20}$$

(b) The direction of max stretch his $\frac{\arg(\mu)}{2}$.

(2) Compute the dilatation for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Now if we define K to be the max dilatation of a quasi-conformal homeomorphism $f : X \to Y$ where X and Y are Riemann surfaces, $K := \sup_{p \in X} K_f(p)$, we can give a definition for Teichmüller distance and state Teichmüller's theorem.

Definition 27. ***Teichmüller distance*** *between any two points $X, Y \in \mathcal{T}(S)$ is given as follows*

$$d_{\mathcal{T}}(X, Y) := \frac{1}{2} \ln \inf_{f : X \to Y} K \tag{21}$$

where each f is a quasi-conformal homeomorphism.

Theorem 28 (Teichmüller's Theorem)**.** *For all $X, Y \in \mathcal{T}(S)$, there exists a unique quasi-conformal $f : X \to Y$ such that $d_{\mathcal{T}}(X, Y) = \frac{1}{2} \ln K_f$.*

These maps described by the theorem are called Teichmüller maps and trace out Teichmüller geodesics in $\mathcal{T}(S)$. Furthermore, the Teichmüller maps are affine in appropriate coordinates (to be described below) and take the form of $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Note that this gives us a dilatation $K = e^{2t}$.

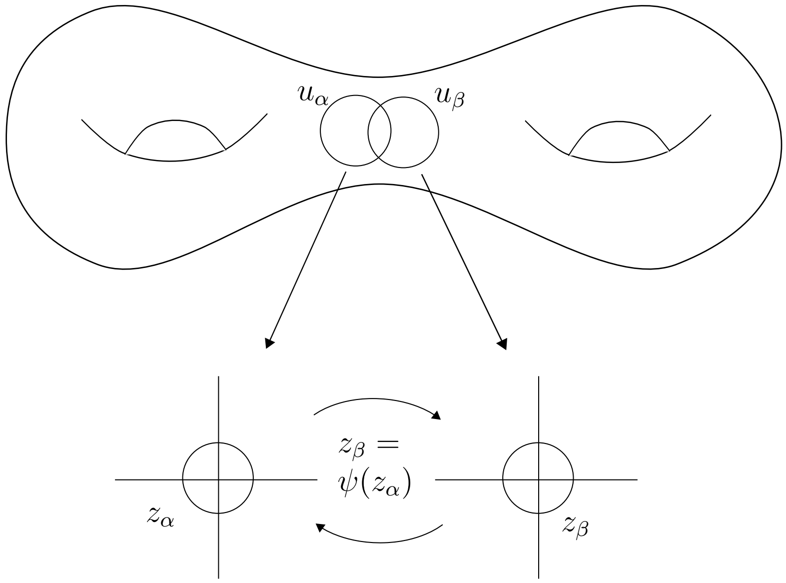

Figure 7: Patches and Transition Function on the Double Torus

6 Quadratic Differentials

(p,q) forms on a Riemann Surface are expressions of the form $\phi(z)dz^p d\bar{z}^q$, meaning holomorphic functions ϕ on each patch of the Riemann Surface with a transformation rule between patches. See figure 7.

Note the following calculation arising from the necessary transformation rule between patches:

$$\phi(z_\alpha)dz_\alpha^p d\bar{z}_\alpha^q = \phi(z_\beta)dz_\beta^p d\bar{z}_\beta^q \tag{22}$$

$$\phi(z_\alpha) = \phi(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^p \left(\frac{d\bar{z}_\beta}{d\bar{z}_\alpha}\right)^q \tag{23}$$

This gives us a means of realizing the holomorphic function ϕ as the derivative of the transition function between z_β and z_α.

Note: Beltrami Differentials, μ_f, are $(-1,1)$ forms.

Definition 29. *An **abelian differential** is characterized by the following equivalent properties:*

(i) A $(1,0)$ *form,*

(ii) a translation structure, i.e. a Riemann surface atlas with translation transitions, or

(iii) or polygonal network with translation gluings.

A ***quadratic differential*** *is characterized by the following equivalent properties:*

(i) A $(2, 0)$ *form,*

(ii) a semi-translation structure, i.e. a Riemann surface atlas with transitions of the form $z \mapsto \pm z + c$ *for some* $c \in \mathbb{C}$*,*

(iii) or a polygonal network with semi-translation gluings.

We can realize a natural topology on $\mathcal{T}(S) = \{\text{marked Poincaré metrics on } S\}$. This topology arises from the following embedding into real space

$$\mathcal{T}(S) \hookrightarrow \mathbb{R}^{\mathcal{S}} \tag{24}$$

$$x \mapsto \{(\alpha, l_x(\alpha))\}_{\alpha \in \mathcal{S}} \tag{25}$$

where $\mathcal{S}$ is the set of simple closed curves on $x \in \mathcal{T}(S)$ and l is the length function.

Teichmüller's Theorem also explains a natural correspondence between the space of quadratic differentials (i.e. polygonal networks or semi-translation surfaces), $Q(S)$, and the tangent space on $\mathcal{T}(S)$. Geodesic paths give the direction vectors, and points are obtained via the forgetful map.

7 Foliations

Abstractly a foliation of a manifold is a partition of the manifold into lower dimensional manifolds called leaves.

Key Example: The Torus

Leaves on the torus are parallel lines. There are two possible cases, either the leaves have rational slope $\frac{p}{q} \in \mathbb{Q}$ in which case they are close, or the slope is irrational and the leaves are equidistributed and dense, although still measure 0. See figure 8.

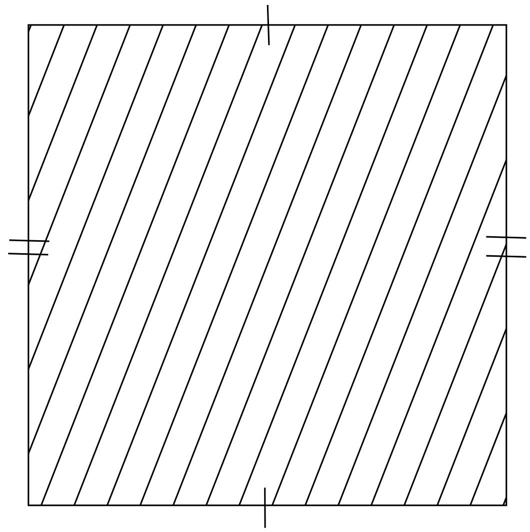

Figure 8: Foliation on the Torus

Definition 30. *A* ***measured foliation*** *on a surface S, denoted $MF(S)$ is a line field in direction θ on a flat structure (either a translation or a semi-translation surface). A* ***measure*** *on a foliation is an assignment of nonnegative real values to arcs on the surface. Here, the measure will be given by the Euclidean distance between leaves.*

See figure 9 for an example of a foliation on the octagon, which becomes a foliation on the double torus if we identify opposite sides.

Now for foliations at cone points we half "k-pronged" singularities which are simply k-half-planes glued together cyclically. See figure 10 for an example with $k = 3$.

Notation: For $\alpha \in \mathcal{S}$ and $F \in MF(S)$ we have $i(\alpha, F)$ = transverse measure of α where α is taken to be the representative of its homotopy class of shortest length.

We realize a topology on $MF(S)$ via an embedding into $\mathbb{R}^{\mathcal{S}}$:

$$MF(S) \hookrightarrow \mathbb{R}^{\mathcal{S}} \tag{26}$$

$$F \mapsto \{(\alpha, i(\alpha, F))\}_{\alpha \in \mathcal{S}} \tag{27}$$

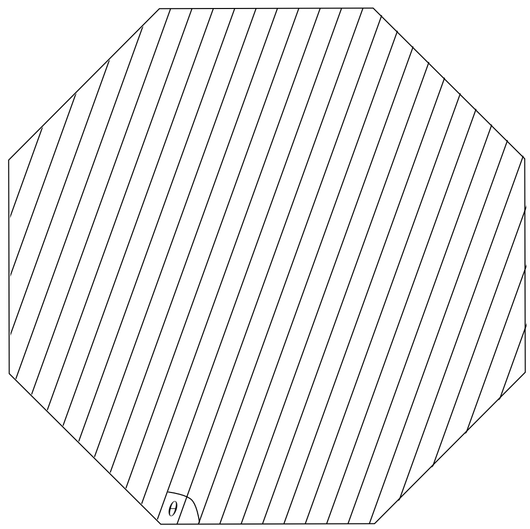

Figure 9: Foliation on the Octagon(i.e. the Double Torus)

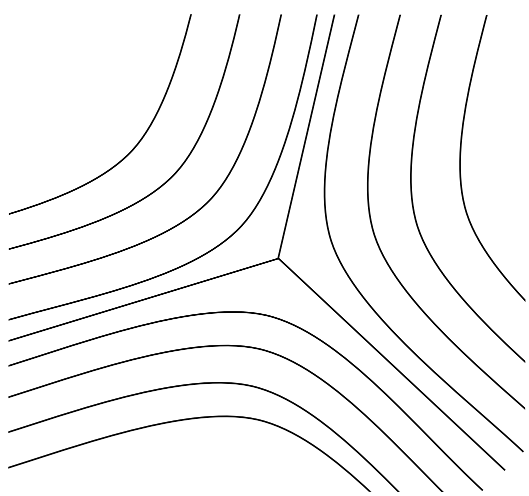

Figure 10: $k = 3$ Pronged Singularity

8 Thurston Compactification of $\mathcal{T}(S)$

We will see that $\mathcal{T}(S) \cong \mathbb{R}^{2\xi}$ and $PMF(S) \cong S^{2\xi-1}$ where $\xi = 3g - 3 + n$ and $PMF(S) = MF(S)/\mathbb{R}^+$. We will define the topology on the compactification via the following convergence criterion:

$$x_n \in \mathcal{T}(S) \to [F] \in PMF(S) \tag{28}$$

$$\text{if} \qquad \frac{l_{x_n}(\alpha)}{l_{x_n}(\beta)} \to \frac{i(\alpha, F)}{i(\beta, F)} \text{ for all } \alpha, \beta \in \mathcal{S} \tag{29}$$

Let's first take a heuristic approach and "watch" a ray leave all compact sets and converge to $PMF(S)$. For a curve $\alpha \in \mathcal{S}$ we can draw α in the flat structure q dual to the geodesic and identify α with the so-called *holonomy vector* giving its horizontal and vertical displacement: $x \leftrightarrow (h_\alpha, v_\alpha)$. Now if we travel along the Teichmüller geodesic by the application of $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ we get

$$\alpha = (h_\alpha, v_\alpha) \xmapsto{g_t} (e^t h_\alpha, e^{-t} v_\alpha) \tag{30}$$

So we have $l_{x_t}(\alpha) = \sqrt{(e^t h_\alpha)^2 + (e^{-t} v_\alpha)^2}$. Thus if $h_\alpha \neq 0$ we have that $l_{x_t}(\alpha) \approx e^t h_\alpha$. Therefore we have the following:

$$\frac{l_{x_t}(\alpha)}{l_{x_t}(\beta)} \to \frac{h_\alpha}{h_\beta} = \frac{i(\alpha, F_{\text{vert}})}{i(\beta, F_{\text{vert}})} \tag{31}$$

The heuristic suggests that a Teichmüller geodesic ray should converge to the vertical foliation of its flat structure.

But it's only a heuristic because the real lengths on the left-hand side are **hyperbolic** lengths, not flat lengths, and in fact it can certainly occur that rays do NOT "hit the target"—some don't converge to any point (Lenzhen) and some converge to points other than their vertical foliations (Masur). It is nonetheless fairly easy to show that every accumulation point of a geodesic ray is disjoint from the vertical foliation: $i(F, F_{\text{vert}}) = 0$.

9 Quadratic Differentials Redux

We saw before that we have the following three definitions for the space of quadratic differentials on a surface S:

$$Q(S) = \{\phi(z)dz^2\} = \{\text{"polygonal networks"}\} = \{\text{semi-translation surfaces}\} \tag{32}$$

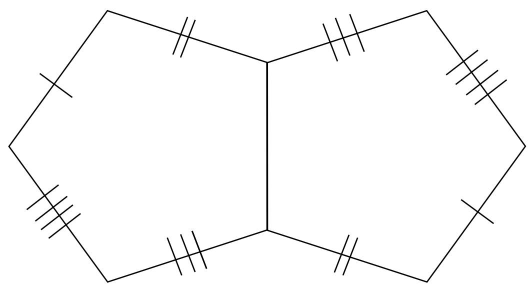

Figure 11: Double 5-gon

Let us give a more concrete definition of polygonal networks.

Definition 31. *A* ***polygonal network*** *is a collection of Euclidean polygons in the plane with side pairings glued by translation or semi-translation. These polygonal networks are considered up to the equivalence of cut and parallel paste.*

Thus we readily see that cone angles are multiplies of π and that polygonal networks have well-defined line fields in a direction θ.

Examples: Regular Polygons:

- For n even we have regular n-gons.
- For n odd we have double n-gons. E.g. figure 11.

Exercise 9.

- *For $n = 5, 8$, realize the n-gon as an L-shaped polygon.*
- *Compute genus of regular n-gon surfaces.*
- *Draw vertical foliation of the octagon topologically on S_2.*

9.1 Slit Tori

We can have foliations of S with some leaves neither closed nor dense, e.g., via the slit torus construction where we have closed leaves on one half of the double torus and leaves that are dense on the other half, see figure 12.

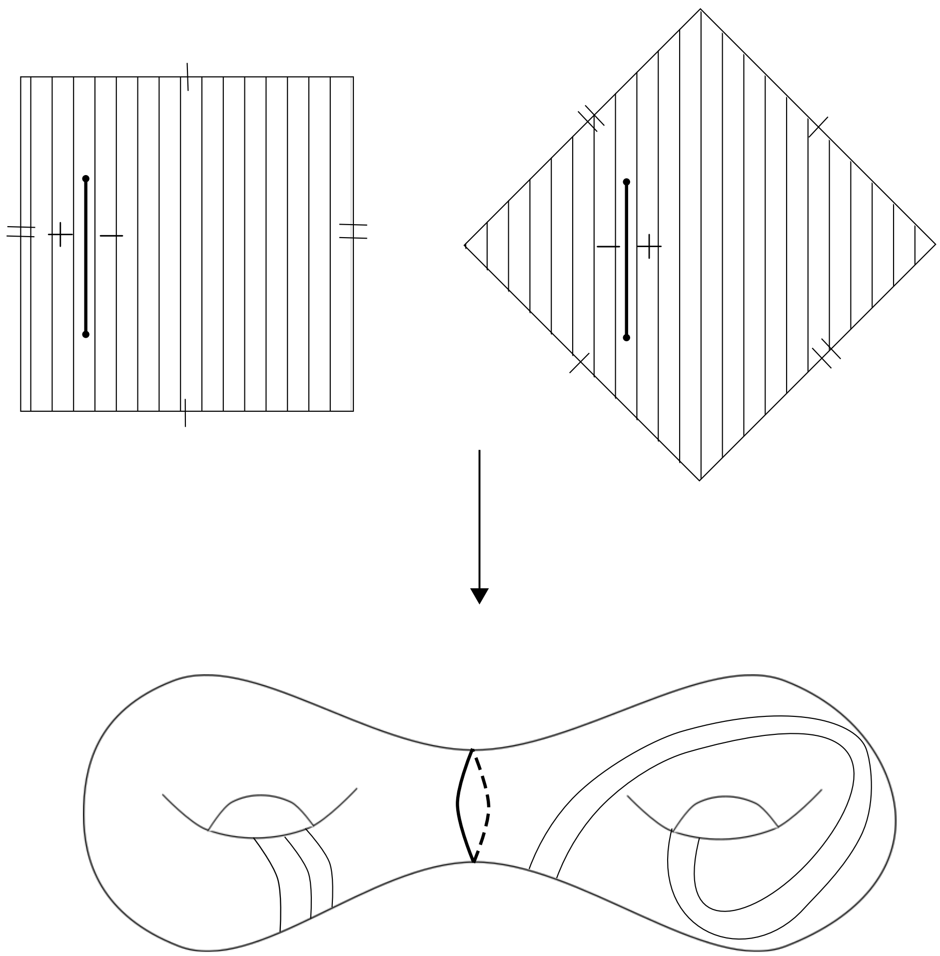

Figure 12: Slit Torus: Leaves are closed on the left side but dense on the right.

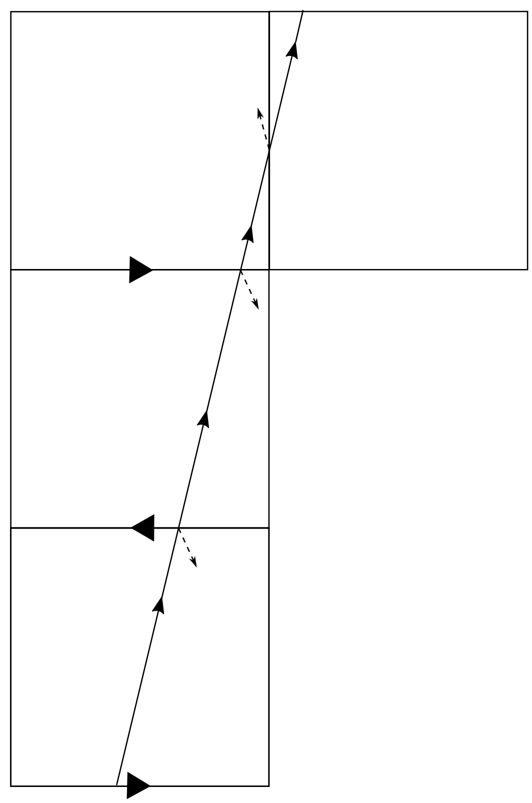

Figure 13: Example of Rational Billiards Development

9.2 Rational Billiards

Rational billiards gives a dynamical system via playing "billiards" on polygons. That is, we begin with a trajectory with angle in $\mathbb{Q}\pi$ and continue traveling in this direction until running into the edge of a polygon and then changing direction via the angle of incidence. However, in order to study the trajectories, instead of changing the trajectory we unfold the billiards table and reflect the polygon in order to keep the trajectory in a straight path.

This gives us a process to create polygonal networks. Given some polygon and angle in $\mathbb{Q}\pi$ we develop the table via reflections until every side is matched with a parallel side in the same direction. See figure 13 for an example of development.

Exercise 10. *Show that the development of a polygonal network via rational billiards terminates in finitely many steps and that the polygonal network is* *independent of choice of angle and starting position (up to parallel cut and paste).*

Note: Every regular polygon is the development of a right triangle with the correct angle, e.g. the octagon is the result of development of a right triangle with an angle of $\frac{\pi}{2}$.

9.3 Proof of Equivalence of QD Definitions

We now prove the equivalence of $\{\phi(z)dz^2 = q\} = \{$"polygonal networks"$\}$.

Proof. We first prove $\{$"polygonal networks"$\} \subseteq \{\phi(z)dz^2 = q\}$. We have a natural chart at every non-cone point of the surface to the complex plane and so we can write $\phi(z_\alpha) = 1$ for every $\alpha \in S$ which is not a cone point. Thus we can write $q = dz_\alpha^2$. Now we can check that this is accurate with respect to the transition functions by recalling the identity $\phi_\alpha(z_\alpha) = \phi_\beta(z_\beta)(\frac{dz_\beta}{dz_\alpha})^2$ and noting that since the transition functions are translation or semi-translations. Therefore, we have $(\frac{dz_\beta}{dz_\alpha})^2$, so we get $\phi_\alpha(z_\alpha) = 1 = \phi_\beta(z_\beta)(\frac{dz_\beta}{dz_\alpha})^2 = 1 \cdot 1$ for non-cone points α and β.

This gives us a quadratic differential at each non-cone point, so all that remains is to see what happens at singularities. Let $\gamma \in S$ be a cone-point, so that we get a k-pronged singularity at z_γ. Let us look at the example of a 6-pronged singularity. We get 6 half-planes, which we write $w_1, w_2, \ldots, w_6$, which are glued together. Note that this is equivalent to the branch point of $z^{1/3}$. Thus we can write $z_\gamma = w_i^{1/3} \Rightarrow z_\gamma^3 = w_i$. Now we can differentiate this to see $dw_i = 3z_\gamma^2$, thus we write $\phi_\gamma(z_\gamma) = 3z_\gamma^2$. See figure 14 for an illustration of this.

It is left to the reader to check that this technique works for any value of k. Thus we have seen that $\{$"polygonal networks"$\} \subseteq \{\phi(z)dz^2 = q\}$.

It remains to be seen that the other inclusion holds, $\{\phi(z)dz^2 = q\} \subseteq \{$"polygonal networks"$\}$. The method that we will use is to build a set of natural coordinates at every point on the surface. Given ϕ_α and a point $z_\alpha \in S$ we perform a change of variables as follows

$$w_\alpha = \int_0^{z_\alpha} \phi_\alpha^{1/2}(\zeta)d\zeta \tag{33}$$

where ζ is a dummy variable. Now by the fundamental theorem of calculas we get

$$\frac{dw_\alpha}{dz_\alpha} = \sqrt{\phi_\alpha(z_\alpha)} \tag{34}$$

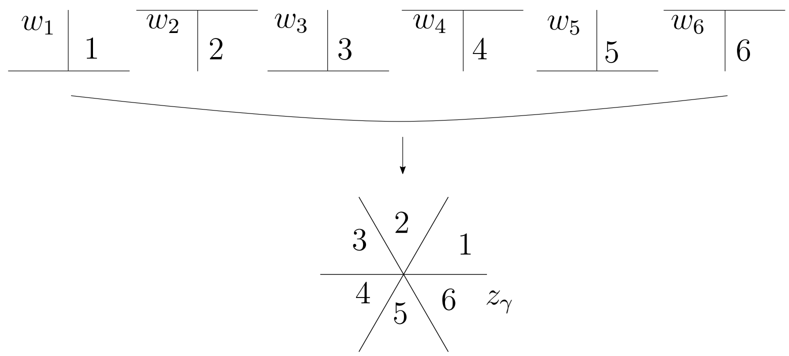

Figure 14: 6-Half Planes Glued Together at a Singularity

So we have

$$dw_\alpha^2 = \phi_\alpha(z_\alpha)dz_\alpha^2 = q \tag{35}$$

Now if we write z_α and z_β as their natural coordinates under this change of variables we get

$$dz_\alpha^2 = dz_\beta^2 \Rightarrow \frac{dz_\beta}{dz_\alpha} = \pm 1 \tag{36}$$

Thus we now have semi-translation transition functions.

Finally to actually realize the corresponding polygonal network we have zeros of q from the quadratic differential away from which we have natural coordinates. To accomplish this we require the following definition.

Definition 32. *A* ***saddle connection*** *is a path on a Riemann surface that is a straight line in natural coordinates connecting two zeros that are not necessarily distinct, but not on its interior.*

Thus to construct the corresponding polygonal network we simply cut q along saddle connections until the pieces are convex polygons. □

10 Teichmüller Space of the Torus

In this section we compute the Teichmüller space and the mapping class group of the torus. By uniformization we know that every torus is the quotient of the complex plane by some lattice, i.e. $\mathbb{T} = \mathbb{C}/\Lambda$ where $\Lambda = \langle \tau_1, \tau_2 \rangle$ for some pair of translations τ_1 and τ_2. Now we note that rotation and dilation are biholomorphic so that we can transform any pair τ_1 and τ_2 so that

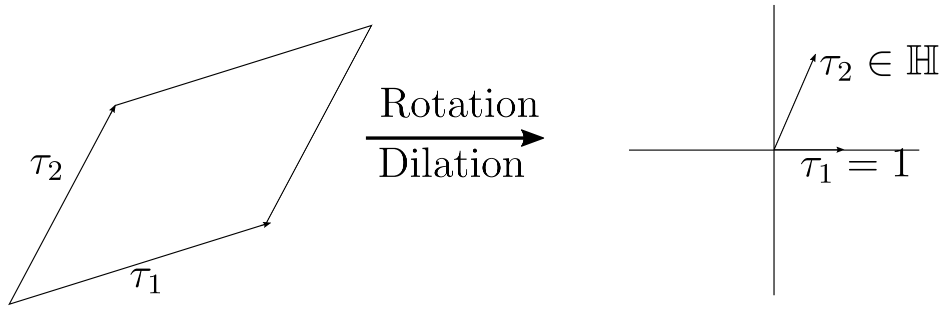

Figure 15: Teichmüller Space of the Torus

$\tau_1 = 1$ and τ_2 is in the upper half plane, i.e. $\tau_2 \in \mathbb{H}$. See figure 15. Therefore we see that $\mathcal{T}(\mathbb{T}) = \mathbb{H}$.

Exercise 11. *Use $\frac{1}{2}\ln K_f$ to measure distance along the imaginary axis.*

Next we compute the mapping class group of $\mathbb{T}$. Recall that $MCG = \langle \text{Dehn Twists} \rangle$. Thus we consider a Dehn twist about $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We see that $T_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Therefore we get $MCG(\mathbb{T}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle \cong SL_2(\mathbb{Z})$.

11 Embedding of PMF into $\mathbb{PR}^{\mathcal{S}}$

Now let us examine further the embedding:

$$PMF \hookrightarrow \mathbb{PR}^{\mathcal{S}} \tag{37}$$

$$F \mapsto \{i(\alpha, F)\}_{\alpha \in \mathcal{S}} \tag{38}$$

We take the point of view that PMF is the abstract completion of $\mathcal{S}$. To realize this we first study the embedding $\mathcal{S} \hookrightarrow PMF$ and realize PMF as the closure of $\mathcal{S}$ with respect to the natural topology obtained from the embedding. Note that we have the following embedding from $\mathcal{S}$ into $\mathbb{PR}^{\mathcal{S}}$ as well

$$\mathcal{S} \hookrightarrow \mathbb{PR}^{\mathcal{S}} \tag{39}$$

$$\gamma \mapsto \{i(\alpha, \gamma)\}_{\alpha \in \mathcal{S}} \tag{40}$$

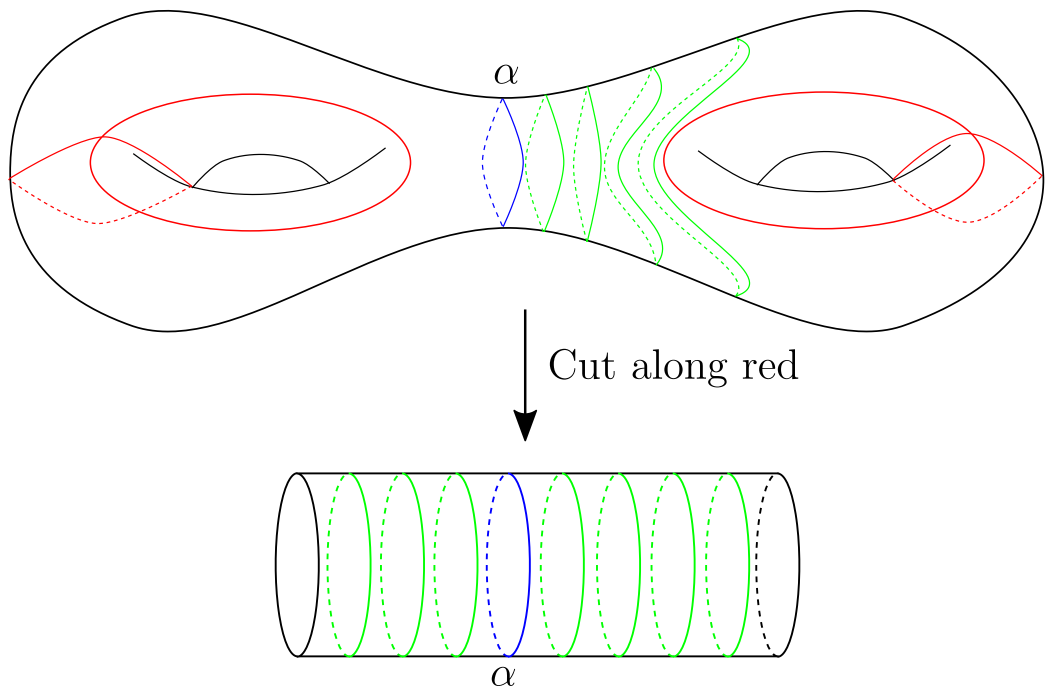

Figure 16: Embedding of curve α into PMF

To realize the embedding of $\mathcal{S}$ into PMF we begin with a curve $\alpha \in \mathcal{S}$ and would like to foliate the surface with homotopic copies of itself. However, this is not possible, so we find the curves which are the minimal obstruction to this and cut along them and obtain a cylinder on which we can find a foliation of homotopic copies of α. Another way to see this is to cut along the meridian and longitude about each genus of the surface to obtain a cylinder. See figure 16 for an example. This gives a topological embedding.

However, what we really desire is an embedding with measure and for that we turn to the topic of cylinder decompositions.

Definition 33. *A* ***cylinder decomposition*** *of a flat structure is a foliation in a direction such that all leaves are closed. A series of cut and parallel pastes may be required to realize this decomposition. See figure 17 for an example on the octagon.*

Note: Not every flat structure has a complete cylinder decomposition, e.g. the slit torus.

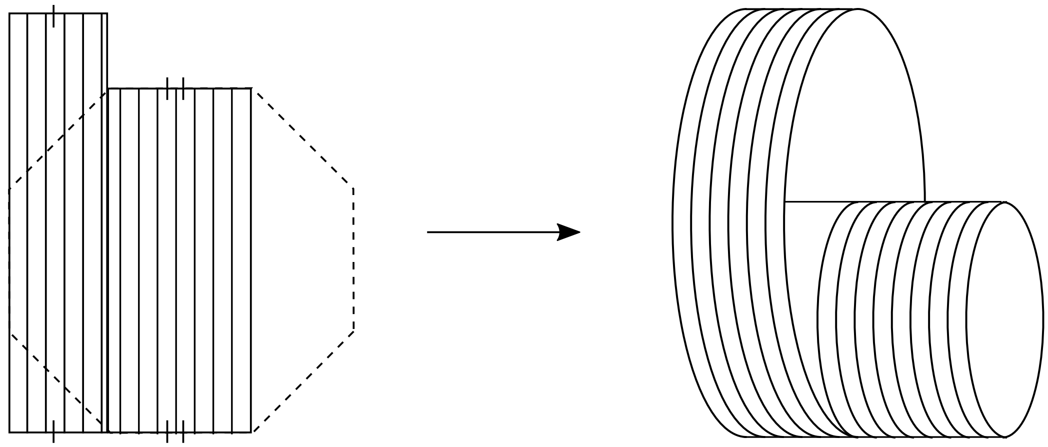

Figure 17: Cylinder Decomposition of the Octagon

Theorem 34 (Masur)**.** *For all quadratic differentials, q, the set of directions with a cylinder is dense.*

Although not every q has a complete cylinder decomposition, it is true that any q decomposes into rectangle, but with possibly more complicated top and bottom gluings. To study these decompositions we use a zippered rectangle construction. First we give the following definition about foliations.

Definition 35. *A foliation is called **minimal** if it has some leaf that is dense on S.*

We initially consider the case of a minimal foliation. Now we draw an arc transverse to leaves. To draw the zippered rectangle we draw the arc horizontally and for each leaf we take the first return to the arc and draw the length vertically. However, we can have leaves which run into a singularity and thus drastically change in length. This is when we introduce the "zippered" part of the construction. The rectangles become unglued at the singularity and separated. Now we can also have different gluings on the ends of the rectangles so that they are no longer simply glued to opposite ends of the same rectangles. See figure 18 for an example.

Note: These zippered rectangle constructions have a connection with the theory of *interval exchange transformations*, which are piecewise isometries of and interval. In fact, they encode the same information as the gluings of the sides of the zippered rectangles. For cylinder decomposition the respective integral exchange transformation is simply the identity.

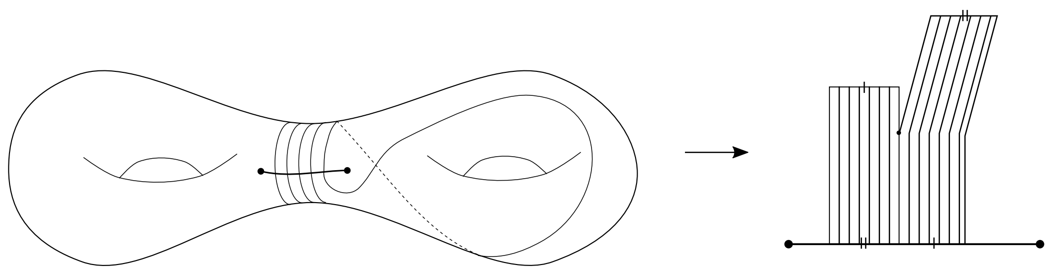

Figure 18: Example of Zippered Rectangle Construction

Theorem 36 (Jenkins-Strebel Theorem)**.** *For any $x \in \mathcal{S}$ there exists a flat structure, q, such that the vertical foliation of q is α, i.e. $F_{vert} = \alpha$*

It is this theorem which gives us the embedding with measure of simple closed curves into PMF.

12 Curve Complex/Curve Graph

For surfaces with $\xi = 3g - 3 + n > 3$ we have the following definition for the curve complex.

Definition 37. *The **curve complex**, $\mathcal{C}$, of a surface S is the graph with a vertex for each simple closed curve on the surface and edges connecting two vertices if the intersection number of the two simple closed curves is 0. That is we have*

$$\begin{aligned} &V(\mathcal{C}) := \mathcal{S} \\ &E(\mathcal{C}) := \{(\alpha, \beta) | i(\alpha, \beta) = 0 \textit{ for } \alpha, \beta \in \mathcal{S}\} \end{aligned}$$

Now we must make a modification for the curve complex for surfaces with $\xi \leq 3$. There are two different possible modifications and we will use the example of the torus to explore these. Indeed, note that on the torus no two curves are disjoint. This leads us to our first modification, denoted $\mathcal{C}'(\mathbb{T})$. For $\mathcal{C}'(\mathbb{T})$ we now take an edge between any $\alpha, \beta \in \mathcal{S}$ whenever $i(\alpha, \beta) = 1$.

Claim: For two curves on the torus, $\alpha = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\beta = \begin{pmatrix} r \\ s \end{pmatrix}$ we have $i(\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix}) \Leftrightarrow \det \begin{pmatrix} p & r \\ q & s \end{pmatrix}$.

The proof of this claim is via the change of coordinates principle. Indeed, given some $\alpha, \beta \in \mathcal{S}$ we can find a homeomorphism f which preserves intersection number and sends $\alpha \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\beta \mapsto \begin{pmatrix} k \\ 1 \end{pmatrix}$ for some $k \in \mathbb{Z}$. However now we have $\det \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = 1$ so that we f can be upgraded to a mapping class, thus realizing f as some $A \in \mathrm{SL}_2(\mathbb{Z})$. Now we can see that we must have

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} p \\ q \end{pmatrix} \tag{41}$$

$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} r \\ s \end{pmatrix} \tag{42}$$

Thus we have $A = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ which, being a mapping class, must have determinant 1. This gives rise to a graph called the Farey graph, see Figure 19, which is the curve graph for the torus

The Farey graph gives continued fractions on the torus. We can draw geodesics through the center of the circle out to the boundary and each time it crosses an edge we record which side has only one vertex on it in order to obtain a cutting sequence of L's and R's. This sequence terminates if and only if the geodesic goes to a point on the boundary, i.e. to a rational number. We can use this cutting sequence to obtain a continued fraction. For some sequence, e.g. $LLRLLLRR\ldots$, we create a continued fraction by counting the lengths of sequences of only L's and only R's. So this sequence above gives the continued fraction $2 + \frac{1}{1+\frac{1}{3+\frac{1}{2+\ldots}}}$. Thus we see again that sequences terminate if and only if they go to points on the boundary, i.e. their corresponding continued fractions terminate to a rational number.

This was the first modification to the curve graph for the torus, there is a second modification called the *curve and arc complex*, denoted $\mathcal{CA}(\mathbb{T})$. To construct the curve and arc complex we look to the punctured torus. Now given simple closed curves α and β we look to find two arcs on the punctured torus α_1 and β_1 which are disjoint from α and β respectively. Note that there is only one such curve on the punctured torus for each simple closed curve on the torus. See Figure 20.

Lemma 38. $i(\alpha, \beta) = 1 \Leftrightarrow i(\alpha_1, \beta_1) = 0$.

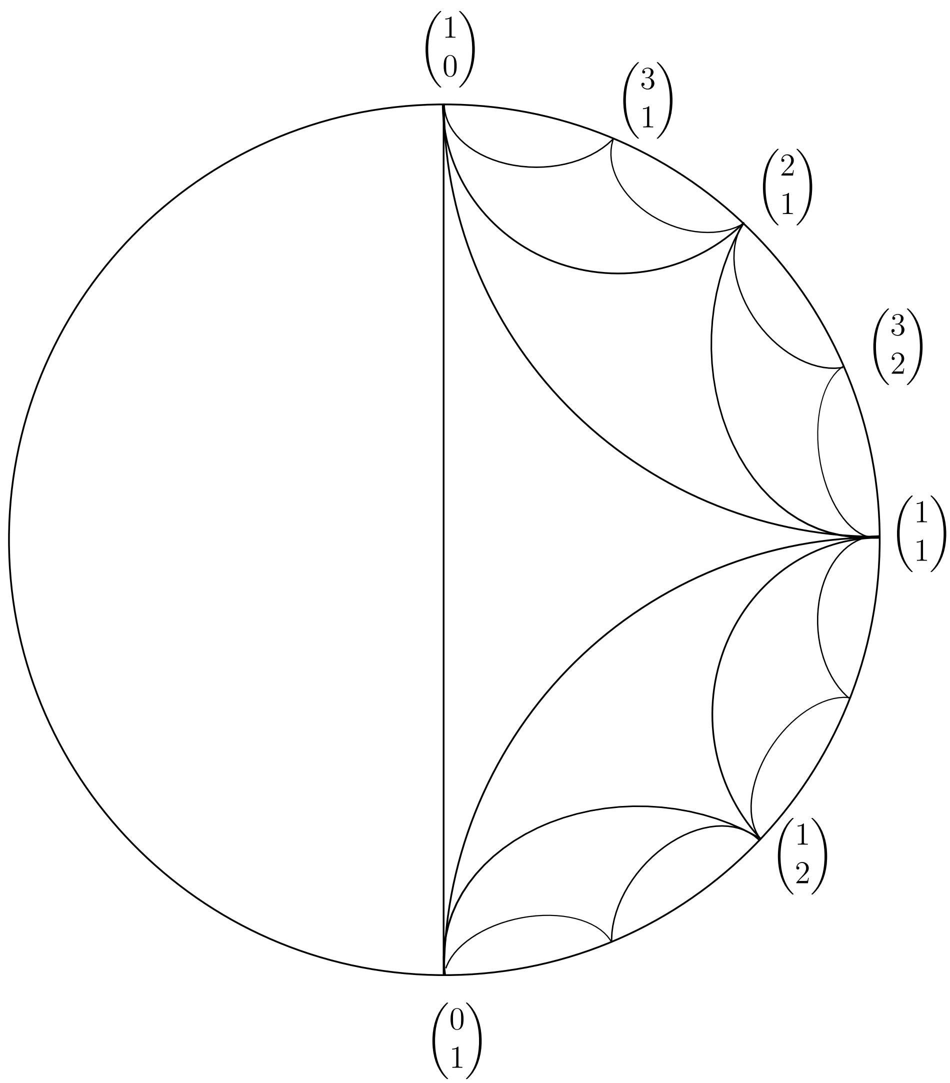

Figure 19: Farey Graph

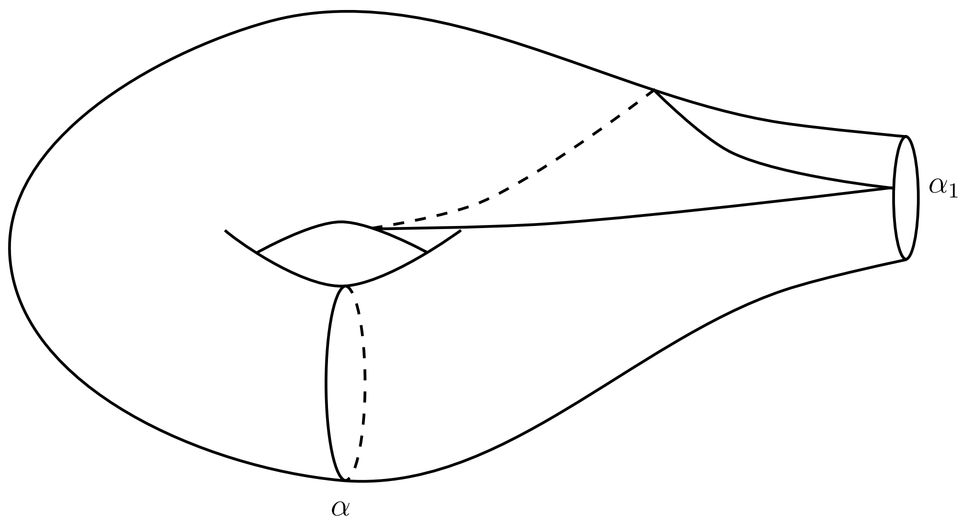

Figure 20: Curves on Punctured Torus for Curve and Arc Complex

Exercise 12. *Prove the above lemma.*

Therefore, in $\mathcal{CA}(\mathbb{T})$ we have

$$\alpha \text{ --- } \alpha_1 \text{ --- } \beta_1 \text{ --- } \beta$$

So $\mathcal{CA}(\mathbb{T})$ is the Farey graph with two vertices inserted on every edge between any two simple closed curves which correspond to α_1 and β_1.

Theorem 39. $\mathcal{C}'(\mathbb{T})$ *is quasi isometric to* $\mathcal{CA}(\mathbb{T})$.

Facts about the Curve Complex:

- $\alpha, \beta \in \mathcal{S}$ fill (i.e. there does not exist some $\gamma \in \mathcal{S}$ such that $i(\alpha, \gamma) = i(\beta, \gamma) = 0$) if and only if $d_{\mathcal{C}}(\alpha, \beta) \geq 3$.
- $\operatorname{diam}(\mathcal{C}) = \infty$.

Other Combinatorial Objects/Graphs Associated to Teichmüller Space:

(i) Curve Graph

(ii) Pants Graph:

- V = pants decompositions
- E = flips of pants

(iii) Marking Graph:

- V = markings, i.e. pants decompositions and transversals intersecting as little as possible.
- Edges consist of three elementary moves, exchanging of pants and transversals, Dehn twists, and half twists.

(iv) Flip Graph:

- V = triangulations of surface
- Edges are flips of diagonals of quadrilaterals.

13 Geodesics on Flat Surfaces

We begin this section with the question of what do geodesics on flat surfaces look like?

Examples: The following are examples of geodesics on flat surfaces:

(1) Cylinder Curves - closed, straight, non-singular curves. To measure distance of cylinder curves, use Euclidean distance on pieces in charts. Thus cylinder curves are geodesics in the pieces on individual charts.

(2) Concatenations of Saddle Connections

(3) Non-example: See Figure 21.

Definition 40. *A curve on a flat surface is said to satisfy the* ***angle condition*** *if it makes angles $\geq \pi$ on each side.*

Now we can state the following proposition characterizing geodesics on flat surfaces.

Proposition 41. *A curve on a flat surface is a geodesic if and only if it satisfies the angle condition.*

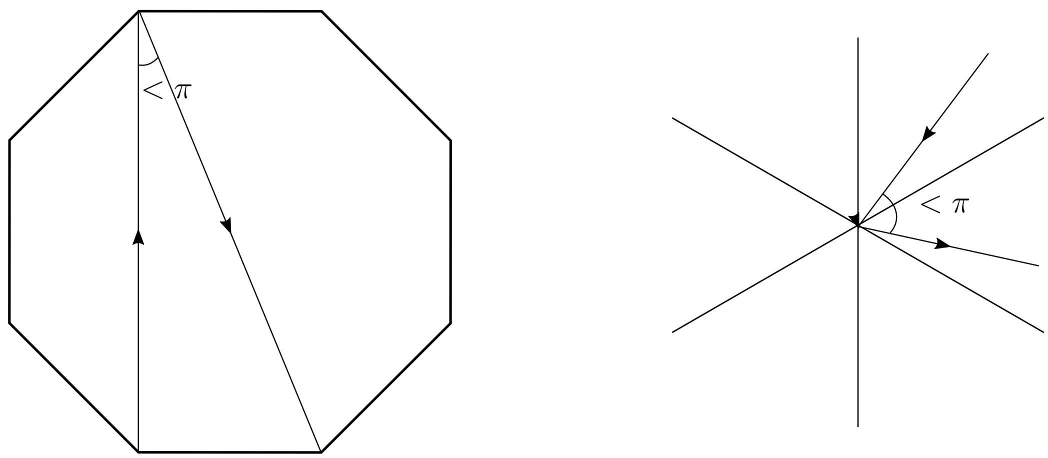

Figure 21: Non-geodesic Curve that Violates the Angle Condition

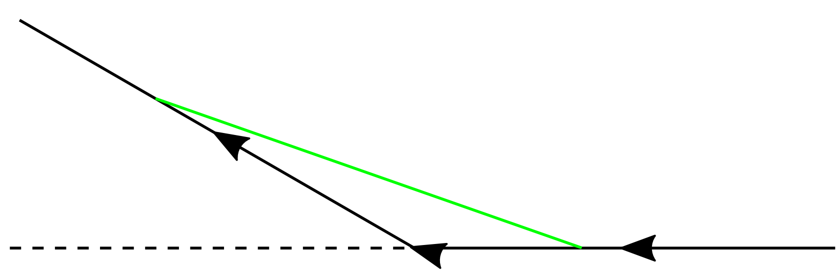

Figure 22: Shorter Representative (Green) for a Curve that does not Satisfy the Angle Condition

Proof. We first show that curves which do not satisfy the angle condition are not geodesics. This is a simple proof by picture, since anytime a curve does not satisfy the angle condition we can find a shorter representative in its homotopy class so that it can not be geodesic, see Figure 22.

We next show that curves which do satisfy the angle condition are geodesic. Let α be a curve on a flat surface q that satisfies the angle condition. Now suppose, to the contrary, that there is a shorter representative for α, i.e. that α is not geodesic. □

14 Length Spectra and Flat Surfaces

We will denote $\lambda(q)$ for the length spectrum of a flat surface q; $\lambda(q) := \{(\alpha, l_q(\alpha))\}_{\alpha\in\mathcal{S}}$.

Theorem 42. *The length spectrum, λ detects the cylinder curves of q, i.e. $\lambda(q) = \lambda(q') \Rightarrow \mathrm{cyl}(q) = \mathrm{cyl}(q')$.*

Proof. Let $\alpha \in \mathrm{cyl}(q)$ and $\beta \in \mathcal{S}$ such that $i(\alpha, \beta) > 0$. We first note that $l_q(T_\alpha(\beta)) < l_q(\beta) + i(\alpha, \beta)l_q(\alpha)$ where T_α is a Dehn twist about α. We can see this readily for the $i(\alpha, \beta) = 1$ case as in Figure 23. Let θ be the angle between α and β, we see that the application of T_α to β gives us the curve obtained from following along β then traveling the length of α back to β. Thus since $\theta < \pi$ we have that the geodesic representative of $T_\alpha(\beta)$ is necessarily shorter.

Next we see that it is also true that for any $\alpha \notin \mathrm{cyl}(q)$ there exists some $\beta \in \mathcal{S}$ such that $i(\alpha, \beta) > 0$ and $l_q(T_\alpha(\beta)) = l_q(\beta) + i(\alpha, \beta)l_q(\alpha)$. To make this argument we must pass to the universal cover of q. We need to find a β which shares a saddle connection with α. This is sufficient because then if we could shorten $T_\alpha(\beta)$ we could also shorten β. The general process for constructing this β is to start with some β and replace it with a highly Dehn twisted curve which will share a piece with α. We make this more precise with the following lemma.

Lemma 43. *For $\alpha, \gamma \in \mathcal{S}$, if $i(\gamma, \alpha) > 0$ then $T^n_\alpha(\gamma)$ follows some saddle connection of α for n sufficiently large.*

Proof. To prove this lemma we lift to the universal cover of q. We can build a geodesically convex neighborhood, $N_{\tilde{\alpha}}$, of the lift $\tilde{\alpha}$ (??). Now we note that the action of T_α on $\tilde{\gamma}$ drags the endpoints of $\tilde{\gamma}$ closer and closer to the

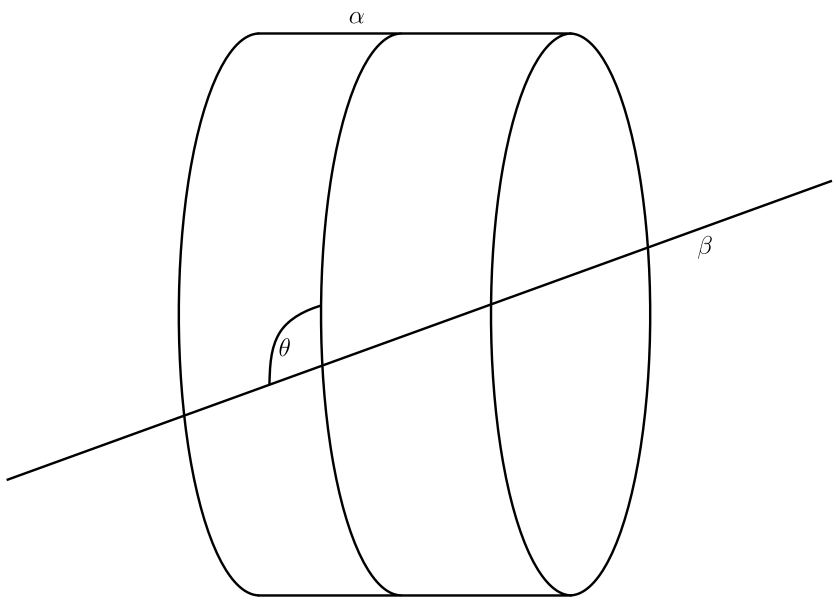

Figure 23: $i(\alpha, \beta) = 1$ Case for Dehn Twist about α

endpoints of $\tilde{\alpha}$. See Figure 24. Thus since $N_{\tilde{\alpha}}$ is geodesically convex we have that for some sufficiently large n, $T^n_\alpha(\gamma)$ must follow some saddle connection of α. $\square$

Finally, say $\alpha \in \mathrm{cyl}(q)$ and consider all β_i with $i(\alpha, \beta_i) > 0$. If all β_i satisfy:

$$l_{q'}(T_\alpha(\beta_i)) < l_{q'}(\beta_i) + i(\alpha, \beta_i) l_{q'}(\alpha) \tag{43}$$

then $\alpha \in \mathrm{cyl}(q')$. (?) $\square$

This above theorem is a specific part of a larger theorem of Duchin, Leininger, and Rafi.

Theorem 44. *If $\lambda(q) = \lambda(q')$ then $q' = e^{i\theta} q$ for some θ.*

15 Pseudo-Anosovs

Definition 45. *A **pseudo-Anosov** mapping class ψ is a map with North-South dynamics on $\mathcal{T}(S)$. That is, there is an attracting fixed point, $\lambda^+ \in$*

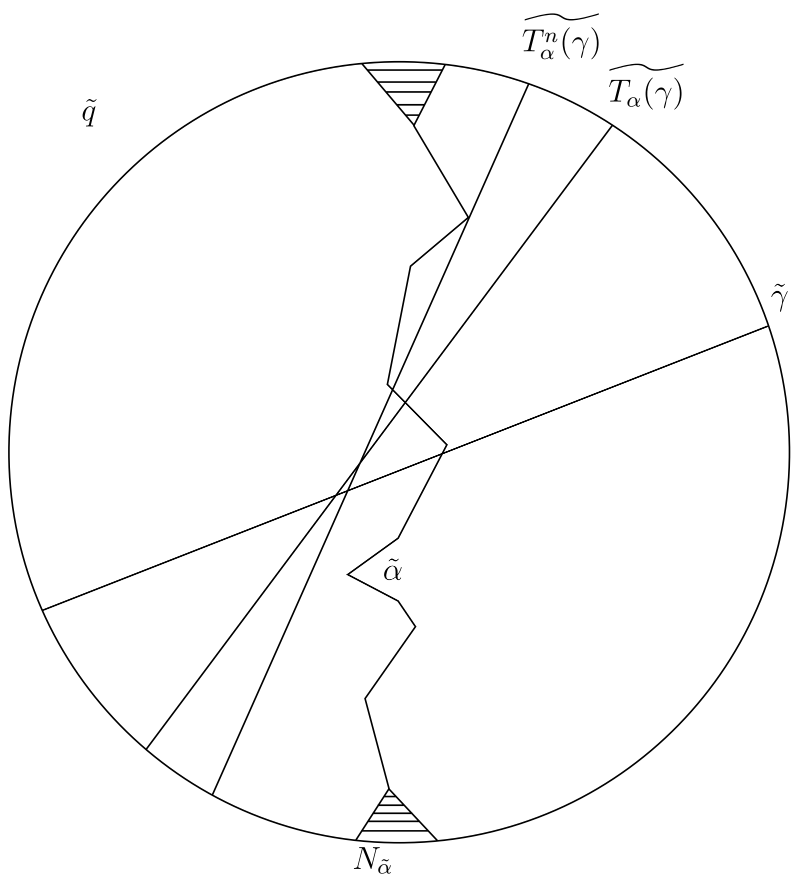

Figure 24: The Action of T_α on γ in the Universal Cover of q

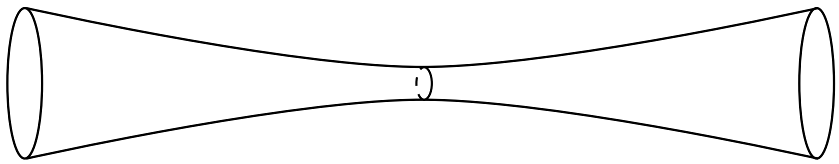

Figure 25: Picture of the Collar Lemma

$\partial\mathcal{T} = PMF$, a repelling fixed point $\lambda^- \in PMF$ and a geodesic axis between them such that points are pushed along the axis, i.e. $\psi^n(x) \to \lambda^+$ and $\psi^{-n}(x) \to \lambda^-$ for all $x \in \mathcal{T}(S)$.

Exercise 13. *Work out the North-South dynamics for $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the torus; i.e., figure out how it acts on the Teichmüller space $\mathcal{T} = \mathbb{H}$ and on the curve complex (the Farey graph). Draw several horocycles for the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ curve and identify $\epsilon - Thin_{(1,0)}$ for various specific values of ϵ. Find an ϵ small enough that this curve stays thick along the axis of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Do the same for the $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ curve.*

16 Length Rigidity

In this section we seek to state and offer some proof of two theorems pertaining to length rigidity of surfaces. We first state a number of results pertaining to hyperbolic geometry and hyperbolic surfaces.

Theorem 46 (Collar Lemma)**.** *Let γ be a simple closed geodesic on a hyperbolic surface X, then $N_\gamma := \{x \in X | d(x,\gamma) \leq \omega\}$ where $\omega = \sinh^{-1}(\frac{1}{\sinh(\frac{1}{2}l(\gamma))})$ is an embedded annulus.*

Note:

(i) As $l(\gamma) \to 0$ we have that $\omega \to \infty$, i.e. short curves have big "collars". See Figure 25.

(ii) The Collar Lemma fails on flat surfaces. To see this we again turn to the slit torus construction. We take a sequence of slit tori $\{q_n\} \subset Q(S)$

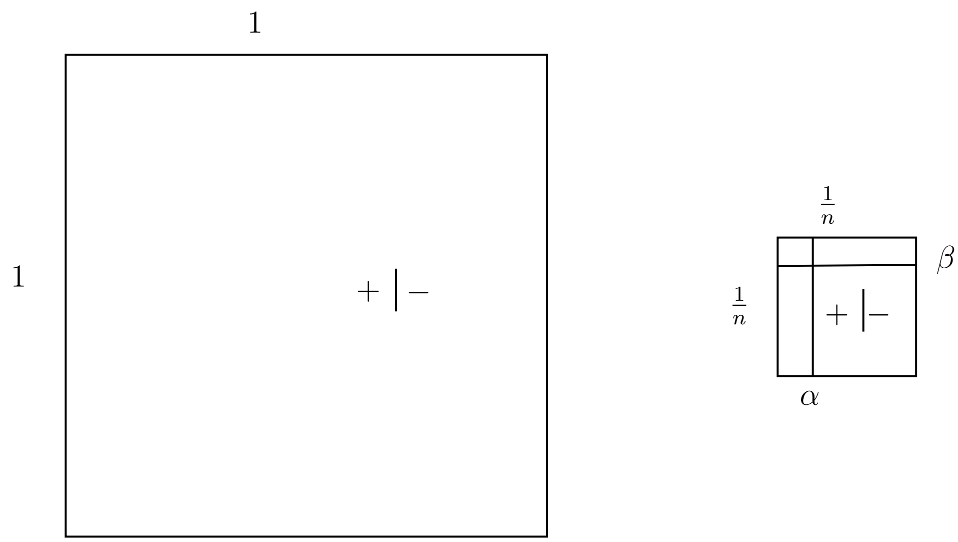

Figure 26: Counterexample to the Collar Lemma for Flat Surfaces: Slit Torus Construction

which consist of one slit square torus of side lengths 1 and another with side lengths of $\frac{1}{n}$ and glue them along their slits, see Figure 26. Now we consider two curves, α and β which are vertical and horizontal curves on the torus with side lengths $\frac{1}{n}$ respectively such that $i_{q_n}(\alpha, \beta)$. Now we have $l_{q_n}(\alpha) = l_{q_n}(\beta) = \frac{1}{n} \to 0$ as n goes to infinity, but $i_{q_n}(\alpha, \beta) = 1$ for all n.

As a corollary to the Collar Lemma we get the existence of the Margulis Constant.

Corollary 47. *(Margulis Constant) There exists $\epsilon_0 > 0$, the **Margulis Constant**, sufficiently small such that no two curves of length less than ϵ_0 can intersect on a hyperbolic surface.*

Thus if $l(\gamma) \leq \epsilon_0$, then for any curve β with $i(\gamma, \beta) > 0$ we must have that $l(\beta) > 2\omega$, or roughly greater than $e^{-\epsilon_0}$. This allows us to state the next corollary pertaining to the existence of the shortest curve, or *systole*.

Definition 48. *The shortest simple closed curve on a surface S is called the **systole**.*

Corollary 49. *The systole on a hyperbolic surface exists.*

Proof. This follows from the fact that there are only finitely many curves shorter than the Margulis Constant, ϵ_0, but only $3g-3$ curves that do not intersect. $\square$

This allows us to create a coarsely well-defined map $\mathcal{T}(S) \to \mathcal{C}(S)$ which maps a point in $\mathcal{T}(S)$ to its systole in $\mathcal{C}(S)$.

Definition 50. *We define the* ϵ**-Thin** *part of Teichmüller space to be:*

$$\epsilon\text{-Thin} := \{x \in \mathcal{T} | l_x(\mathit{systole}) < \epsilon\} \tag{44}$$

Note: If $\epsilon < \epsilon_0$ then,

(i) If for $\alpha, \beta \in \mathcal{S}$, $i(\alpha, \beta) > 0$ then $\epsilon\text{-Thin}_{\{\alpha,\beta\}} = \varnothing$.

(ii) Consider $\Gamma = \{\gamma_1, \ldots, \gamma_k\} \subset \mathcal{S}$. If $k > \xi$ then $\epsilon\text{-Thin}_\Gamma = \varnothing$.

Definition 51. *Let S be a surface with metric ρ and $\Sigma \subset \mathcal{S}$. The **length spectrum** of Σ with respect to ρ is defined to be*

$$\lambda_\Sigma(\rho) := \{\sigma, l_\rho(\sigma)\}_{\sigma \in \Sigma} \tag{45}$$

*We say that $\Sigma \subset \mathcal{S}$ is **length spectrally rigid** over the class of metrics on S if $\lambda_\Sigma(\rho) = \lambda_\sigma(\rho')$ implies that ρ is isometric to ρ' for all metrics ρ and ρ' on S.*

We can now state the two major theorems pertaining to length spectral rigidity.

Theorem 52 (Hyperbolic Length Rigidity, Fricke)**.** *For hyperbolic surfaces, $9g-9$ curves suffice for length spectral rigidity.*

Note: Hamenstädt showed that for hyperbolic surfaces the optimal number of curves for length spectral rigidity in $6g-5$.

Theorem 53 (Flat Length Rigidity)**.** *$\mathcal{S}$ is spectrally rigid over $Q^1(S)$, unit area flat structures, but no finite subsets suffice. Also, $\Sigma \subset \mathcal{S}$ is spectrally rigid over $Q^1(S)$ in and only if $\bar{\Sigma} = PMF$.*

Before proceeding to the proof of these two theorems we first state two other theorems pertaining to length spectra that may be of interest.

Theorem 54 (Birman-Series)**.** *On a hyperbolic surface the set of all simple closed geodesics is nowhere dense on the surface.*

Theorem 55 (Smillie-Vogtman)**.** *The set of conjugacy classes of words in F_n is spectrally rigid over* $\mathrm{Out}(F_n)$ *but no finite set suffices.*

Now we offer a proof of hyperbolic length rigidity. First we state a definition and black box theorem that will be used.

Definition 56. *A function $f : \mathbb{R} \to \mathbb{R}$ is **strictly convex** if for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$ we have*

$$f(tx + (1 - t)y) < tf(x) + (1 - t)f(y) \tag{46}$$

Exercise 14. *Show that the sum of two strictly convex functions is strictly convex.*

Theorem 57 (Black Box 1)**.** *Given a simple closed curve γ on a surface S, let x_t be the path in $\mathcal{T}(S)$ defined by letting the twist parameter of γ be $t \in \mathbb{R}$ with all other coordinates fixed. Then $l_{x_t}(\beta)$ is strictly convex for all $\beta \in \mathcal{S}$ such that $i(\beta, \gamma) > 0$. In other words, "twisting paths are length-convex".*

Hyperbolic Length Rigidity, Fricke. Let S_g be a hyperbolic surface and $\gamma_1, \ldots, \gamma_{3g-3}$ be pants curves. Choose $\beta_1, \ldots, \beta_{3g-3}$ such that $i(\beta_i, \gamma_i) > 0$ and $i(\beta_i, \gamma_j) = 0$ for all $i \neq j$. Let $\alpha_i = T_{\gamma_i}(\beta_i)$, β_i after being Dehn twisted about γ_i. We seek to show that $\{\alpha, \beta, \gamma\}$ is spectrally rigid over $\mathcal{T}(S_g)$.

It suffices to show that if $l_X(\gamma_i) = l_{X'}(\gamma_i)$ for all i but twists about γ_i are different then either some $l_X(\alpha_i) \neq l_{X'}(\alpha_i)$ or $l_X(\beta_i) \neq l_{X'}(\beta_i$. Let $\vec{\tau} := (\tau_1, \ldots, \tau_{3g-3})$ be the vector of twist parameters. Then $X_{\vec{\tau}}$ is a locus in $\mathcal{T}(S_g)$. Now let X_t be the twisting path about γ_i, $A(t) = l_{X_t}(\alpha_i)$ and $B(t) = l_{X_t}(\beta_i)$. These two functions are strictly convex by the black box above, as is their sum, $A(t) + B(t)$ by the exercise above. Note that $A(t + 2\pi) = B(t)$ by the definition of α_i and β_i. Now suppose that $A(s) = A(0)$, say $s > 0$ as in Figure 27. We claim that $B(0) = A(2\pi) \neq A(2\pi + s) = B(s)$. To prove this we consider 3 possible cases.

Case 1: If $s < 2\pi$ then $s < 2\pi < 2\pi + s$ which implies that $A(2\pi + s) > A(2\pi)$.

Case 2: If $s > 2\pi$ then $0 < 2\pi < s < 2\pi + s$ which implies that $A(2\pi + s) > A(2\pi)$.

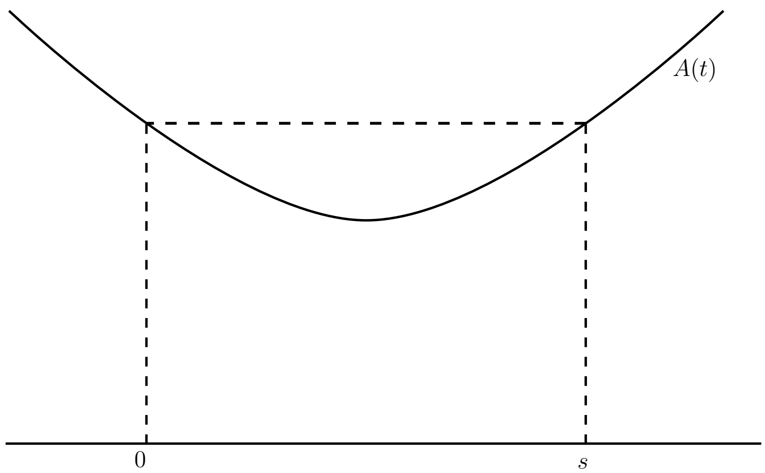

Figure 27: Strict Convexity of $A(t)$

Case 3: If $s = 2\pi$ then, suppose, to the contrary, that $A(2\pi) = A(2\pi + s) = A(4\pi)$. Then $A(0) = A(2\pi) = A(4\pi)$ which contradicts the strict convexity of $A(t)$.

These three cases prove the above claim and finish the proof of the theorem. $\square$

Now we state three more black box theorems before turning to the flat surface case.

Theorem 58 (Black Box 2, Masur)**.** *Given a flat structure q, the cylinder directions on q are dense.*

Theorem 59 (Black Box 3, Thurston)**.** $i(\cdot, \cdot)$ *is continuous on $MF \times MF$.*

Theorem 60 (Black Box 4, Masur)**.** *Uniquely ergodic foliations, $\mathcal{UE} \subset PMF$, are those foliations that topologically only support one measure. They have full measure.*

Proof. (Flat Surface Rigidity)

Let q and q' be two flat surfaces with $\lambda_{\mathcal{S}}(q) = \lambda_{\mathcal{S}}(q')$. We lay out the proof that $q \cong q'$ in three stages. First we show that $\mathrm{PMF}(q) \cap \mathcal{UE} = \mathrm{PMF}(q') \cap \mathcal{UE}$. Then we show this implies that $\mathbb{H}_q = \mathbb{H}_{q'}$. Finally we conclude that q is isometric to q'.

INSERT PROOF $\square$

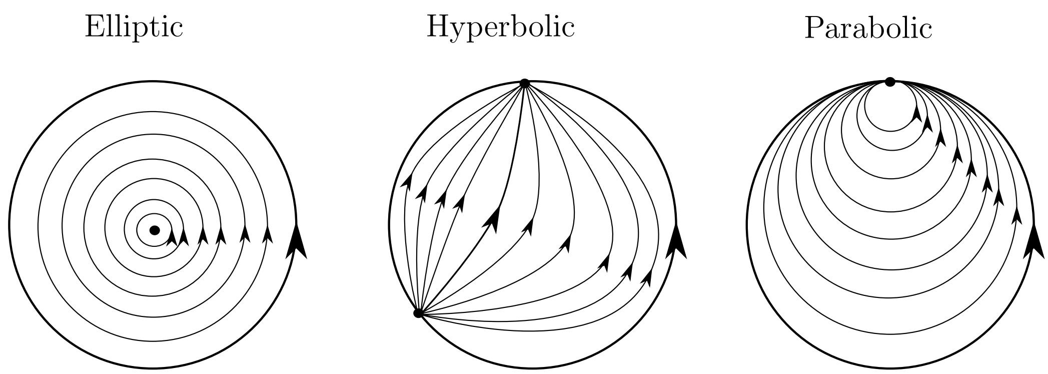

Figure 28: Examples of Elliptic, Hyperbolic, and Parabolic Elements of $\mathrm{SL}_2\mathbb{R}$

17 Classification of Mapping Classes

Recall the classification of the action of $\mathrm{SL}_2\mathbb{R}$ on $\mathbb{H}$. Elements of $\mathrm{SL}_2\mathbb{R}$ can be classified by their trace as follows, see Figure 28:

- if $|\mathrm{tr}| > 2$ then the element is hyperbolic and has two fixed points on $\partial\mathbb{H}$,
- if $|\mathrm{tr}| = 2$ then the element is parabolic and has one fixed point on $\partial\mathbb{H}$,
- and if $|\mathrm{tr}| < 2$ then the element is elliptic and has one fixed point in $\mathbb{H}$.

We now introduce another means of classification.

Definition 61. *The* ***translation length*** *of a map $f : X \to X$ (for X a metric space) is defined to be $\tau_f := \inf_{x \in X} d(x, f(x))$, i.e. the least distance that any point in X is moved.*

We can make the following definitions which are in line with the above classification of $\mathrm{SL}_2\mathbb{R}$.

Definition 62. *Let X be a metric space and $f : X \to X$ a map. If*

1. *$\tau > 0$ and achieved, then f is* ***hyperbolic****,*
2. *τ is not achieved, then f is* ***parabolic****,*
3. *$\tau = 0$ and achieved, then f is* ***elliptic****.*

We turn now to the question: how do Mapping Classes act on $\mathcal{T}(S)$? First recall the following definitions of the Mapping Class Group and Teichmüller space.

$$\mathrm{MCG}(S) = \langle \text{Dehn Twists} \rangle = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S) \tag{47}$$

I.e. the Mapping Class Group is generated by Dehn twists of the surface and is the collection of classes of orientation preserving homeomorphisms up to isotopy.

$$\begin{aligned} \mathcal{T}(S) = \{(X, \phi) | X & \text{ Poincare hyperbolic metric on } S, \\ & \phi : S_0 \to X \text{ homeomorphism}\} \end{aligned} \tag{48}$$

Now we can see that the action of the $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$ is:

$$f \in \mathrm{MCG}, \quad f(X, \phi) = (X, \phi \circ f) \tag{49}$$

Also recall that $\mathcal{T}(S)/\mathrm{MCG}(S) = \mathcal{M}(S)$, the Moduli Space of the surface, which informally is the space of only metrics which forgets markings. Thus we see that mapping classes do not change metrics on the surface.

Example:

Recall that the Teichmüller space of the torus is $\mathbb{H}$ and $\mathrm{MCG}(\mathbb{T}) = \mathrm{SL}_2\mathbb{Z}$. The moduli space for the torus, $\mathcal{M}(\mathbb{T})$, is the modular surface obtained by gluing up the fundamental domain for the action of $\mathrm{SL}_2\mathbb{Z}$. See Figure 29. In the case of the torus the moduli space is an orbifold with 2 cone points. The thin part is a cusp and the thick part is compact.

Next we ask the question, where are curves short? Which leads to the thick-thin decomposition of Teichmüller space. We first consider the example of the torus. Note that since we are renormalizing to area 1 as we travel up the i-axis the systole gets shorter. Then we tile by the fundamental domain. See Figure 30.

Note: The thick part is no longer compact in Teichmüller space.

Exercise 15. *We'll prove that there is a Bers constant: There exists $L_B = L_B(S)$ such that for all $x \in \mathcal{T}(S)$, there is a pants decomposition all of whose curves have length at most L_B.*

(a) What is the area of S? (this is background: google "Gauss-Bonnet" if you don't know the answer.)

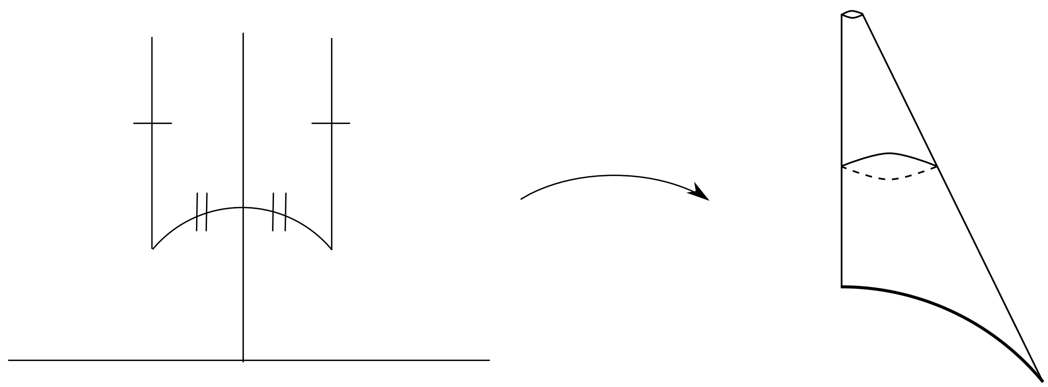

Figure 29: Folding of the Fundamental Domain of the $\mathrm{SL}_2\mathbb{Z}$ Action to $\mathcal{M}(\mathbb{T})$

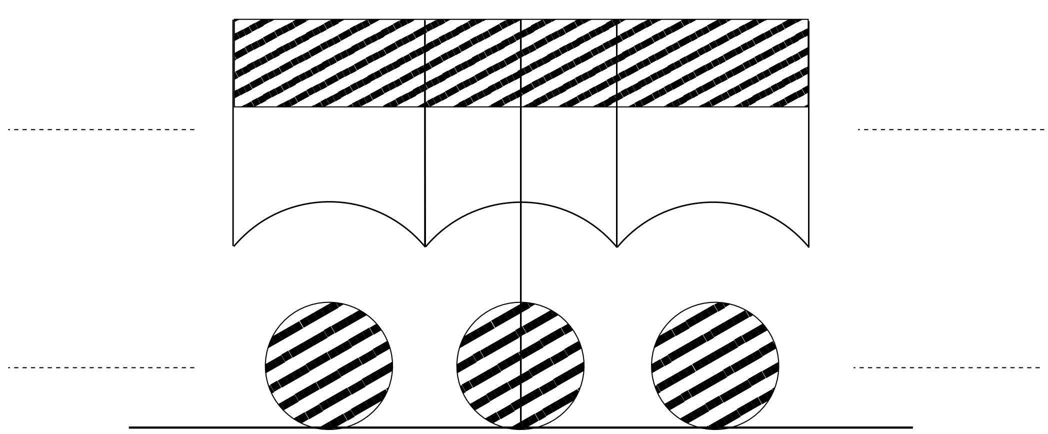

Figure 30: Thin Part of $\mathcal{T}(\mathbb{T})$

(b) *Suppose Y is a hyperbolic (Poincaré) surface with totally geodesic boundary. Consider a largest (open) embedded disk $B = B_r(x_0)$ in Y. Explain why one exists. Either ∂B is self-touching or it touches ∂Y at least twice. Explain.*

(c) *Given an arc α between (not necessarily distinct) boundary components β_1 and β_2 of an annulus, consider a metric neighborhood of $\alpha \cup \beta_1 \cup \beta_2$. Its boundary consists of null-homotopic curves. How about an arc on a pair of pants? How about on a surface more complex than a pair of pants?*

(d) *Prove the existence of a Bers constant.*

We now state the first result pertaining to the thick-thin decomposition.

Theorem 63 (Mumford Compactness Criterion)**.** *ϵ-Thick part of $\mathcal{M}(S)$ is compact.*

Proof. (Sketch)

We show that the ϵ-thick part of $\mathcal{M}(S)$ is sequentially compact. Consider the covering map $\pi : \mathcal{T}(S) \to \mathcal{M}(S)$. Let $\{x_i\}$ and $\{\pi(x_i)\}$ be sequences in $\mathcal{T}(S)$ and $\mathcal{M}(S)$ respectively. We want to show that $\{\pi(x_i)\}$ subconverge and that the pre-image of the subsequence enters a closed and bounded region in $\mathbb{R}^{6g-6}$ (FN-coordinates). Now take Bers pants, there curves have length $l \in [\epsilon, L_B]$. Via the pigeonhole property we can pass to a subsequence with all equivalent Bers pants decompositions. This gives bounds on all the lengths or curves. Now we can also bound the twist parameters, $\tau \in [0, 2\pi]$ via Dehn twisting. Therefore, the subsequence lives in a closed and bounded region in $\mathbb{R}^{6g-6}$, specifically $[\epsilon, L_B]^{3g-3} \times [0, 2\pi]^{3g-3}$, and so converges. □

Exercise 16. *Alexander method: Given an action of $f \in MCG$ on a filling gamily of curves, f is "basically" determined by this action. This uses the fact (Alexander Lemma) that the mapping class group of a disk is trivial. Try to make the statement precise before looking up.*

Theorem 64 (Wolpert's Lemma)**.** *For $X_1, X_2 \in \mathcal{T}(S)$ we have*

$$\frac{1}{K} l_{X_2}(\alpha) \leq l_{X_1}(\alpha) \leq K l_{X_2}(\alpha) \tag{50}$$

for all $\alpha \in \mathcal{S}$ and where $\frac{1}{2} \ln K = d_{\mathcal{T}}(X_1, X_2)$.

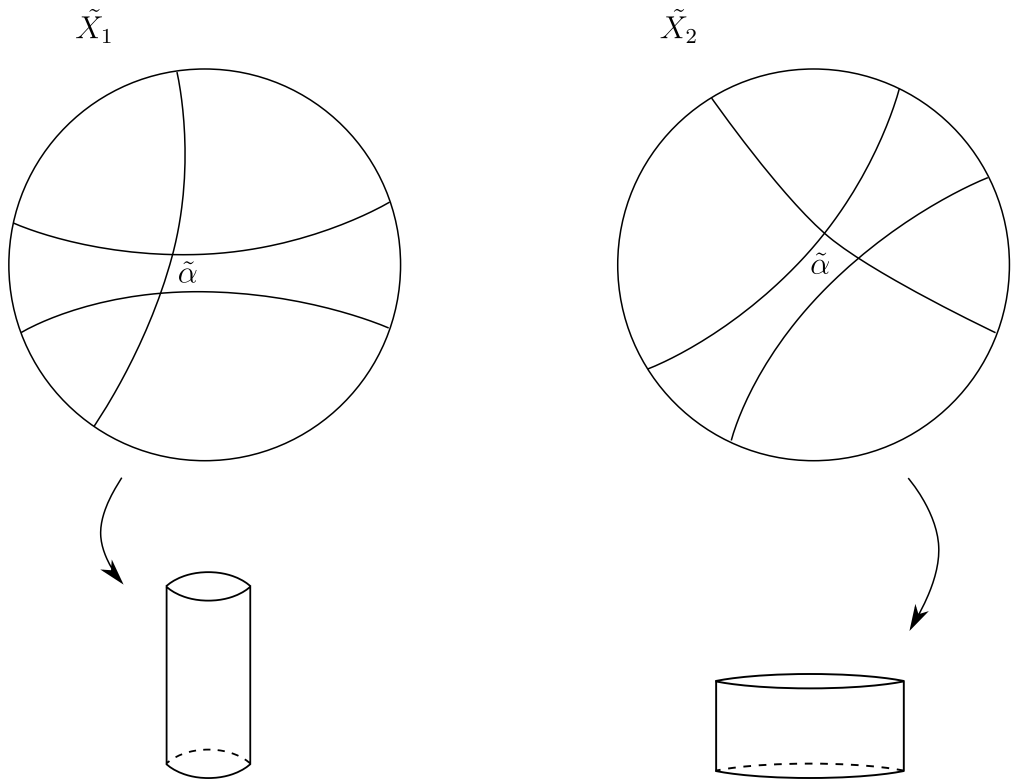

Figure 31: Lifts of α and Corresponding Quotient Annuli

Proof. (Sketch)

This proof uses no Teichmüller theory, only hyperbolic and Euclidean geometry. We look first at the lifts of α in the universal cover of X_1 and X_2. We now quotient by deck transformations to obtain two hyperbolic annuli. Now there are two unique Euclidean annuli which are biholomorphic to these hyperbolic annuli. See Figure 31. (?) $\square$

Definition 65. *Let X be a topological space, K a compact subset and G a group acting on X. Then the action of G on X is said to be* ***properly discontinuous*** *if the set $\{f \in G | f(K) \cap K \neq \varnothing\}$ is finite.*

Proposition 66. *The action of the Mapping Class Group on Teichmüller space is properly discontinuous.*

Proof. **FILL IN LATER** $\square$

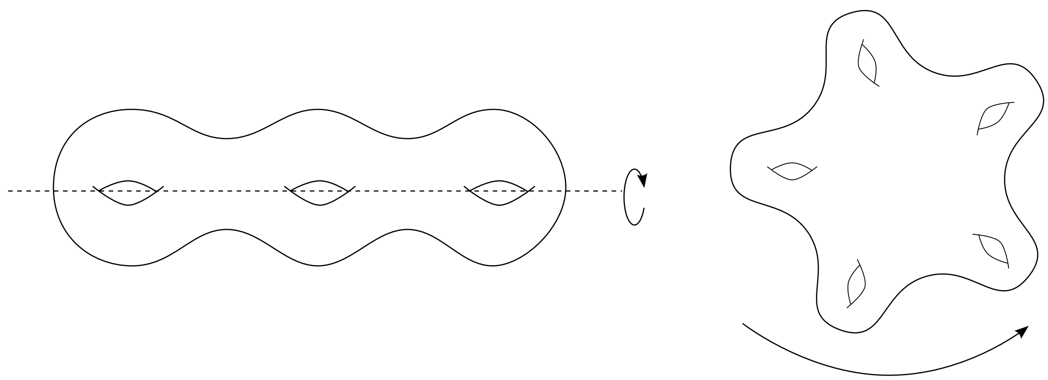

Figure 32: Examples of Finite Order and Reducible Mapping Classes

Theorem 67 (Nielsen-Thurston Classification Theorem for MCG)**.** *Elements, f, of the Mapping Class Group can be classified as:*

- *Finite order,*
- *reducible, i.e. f fixes some multicurve on the surface,*
- *or pseudo-Anosov, i.e. there exists a pair of fixed points on the boundary of Tiechmüller space, one attracting and one repelling.*

Note: A mapping class can be both finite-order and reducible but not pseudo-Anosov and finite-order or reducible.

Examples:

- The action of $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ on the torus is pseudo-Anosov.
- Hyperelliplic involutions ("flip the kebab") are both finite order and reducible, see Figure 32.
- Rotations of symmetric surfaces via the appropriate multiples of π are also both finite order and reducible, see Figure 32.
- Dehn twists are reducible but not finite order.
- Rotations in the fundamental domain of a surface are finite order but not reducible.

Recall that we classified $\mathrm{Isom}^+(\mathbb{H})$ in terms of the translation length of elements into elliptic, parabolic, and hyperbolic elements. We begin our classification of mapping classes via observing the behavior of elliptic, parabolic, and hyperbolic elements of the mapping class group. For $f \in \mathrm{MCG}(S)$, translation length is defined analogously as $\tau_f := \inf_{X \in \mathfrak{T}(S)} d_{\mathfrak{T}}(X, f(X))$. We begin with the elliptic case.

Proposition 68. *If $f \in MCG(S)$ is elliptic then it is finite order.*

Proof. Recall that f is elliptic if $\tau_f = 0$ and realized. However, this implies that there exists a fixed point $X \in \mathfrak{T}(S)$ such that $f(X) = X$. Therefore, f is finite order. $\square$

Proposition 69. *If $f \in MCG(S)$ is parabolic then it is reducible.*

Proof. We seek to show that if τ_f is not achieved we can find a reducing systems (a setwise fixed multicurve). **FINISH** $\square$